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THE FUNDAMENTAL THEOREM OF ELECTRICAL NETWORKS*

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1. Introduction. This paper has been written to meet a need which I believe is real. The theory of electrical networks involves questions of topology, and electrical engineers cannot be expected to be expert topologists. They need only a little, but it is difficult to get to know anything at all about topology without prolonged concentrated study, for it is a closely-knit subject. Hence the necessity to pull out of the body of topology and exhibit in what the mathematician might consider a clumsy popular form, those theorems which are of fundamental importance in network theory.

Of such theorems there appears to be one outstanding, and to it this paper is devoted. Without an understanding of this theorem, every electrician who mixes mesh methods with branch methods must feel insecure; with an understanding of it, he should be able to see the essential simplicity of much that is otherwise obscure and difficult.

The thought that the present attempt must be made, even by one who is neither topologist nor electrical engineer, came to me when reading for review P. Le Corbeiller's book, *Matrix Analysis of Electrical Networks* (Harvard University Press, 1950). There the author avoids the essential topological issues, but it seemed to me that if only those issues could be discussed and understood, the whole significance of Kron's method as expounded in the book would stand out more clearly.

The matter here presented is based on the work of W. H. Ingram and C. M. Cramlet[1], expanded in some respects to make the argument easier to follow but with omission of all that does not seem to bear directly on the fundamental question.

Before proceeding to the technical arguments, I have inserted in the next section some philosophical ideas which may be obvious, but which I think need to be stated with a view to better understanding between mathematicians on the one hand and physicists and engineers on the other.

2. The meaning of "proof". The modern meaning of the words "mathematical proof" is well known: they imply a faultless logical chain which starts from undefined elements and axioms, no loop-holes or exceptions permitted. Those subjects which, like topology, have been developed in that spirit of rigor, present an almost impenetrable front to applied mathematicians or engineers. The extraction of one needed result, and an understanding of what it means, demand a long and careful study in an uncongenial atmosphere, unrelieved by interpretations in terms of natural phenomena.

It is obvious that this state of affairs cannot persist. Each mathematical subject must be treated on several levels, varying from that of extreme rigor down to simple intuitive

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descriptions with no proof at all. We get this variety of treatment in older branches of mathematics, partly because the creators had not got the modern standards of rigor, and partly because these matters have been looked at so long by so many people and from so many different angles.

Much of the argument that passes for proof in physics and engineering is not proof in the mathematical sense, and it is unlikely that the practical subjects will ever submit to the strict mathematical discipline. There seems to be an incompatibility between those minds which excel in logic and those which are capable of dealing successfully with problems suggested by nature.

The word "proof" has such a general usage that it is inconceivable that it should be employed only in its strictest mathematical meaning. Physicists and engineers will continue to use it in a looser sense, and will regard as proved any proposition with regard to

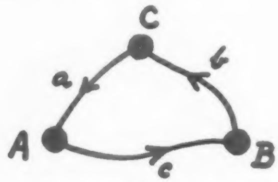

FIG. 1. Network with 3 nodes and 3 branches ($N = 3$, $B = 3$).

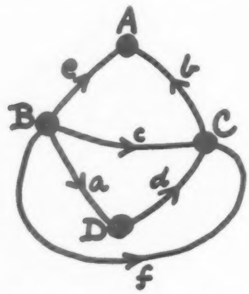

FIG. 2. Network with 4 nodes and 6 branches ($N = 4$, $B = 6$).

which they can assemble sufficient evidence to convince them of its truth. As Descartes pointed out long ago, to "prove" something by a series of logical steps, and to "see" or "understand" it are not the same thing; and what the physicist or engineer needs is the "seeing" and the "understanding".

The type of proof used in the present paper may appear clumsy, longwinded, and inaccurate to the pure mathematician. But that does not matter provided the proofs fulfil their purpose, which is to carry conviction to those who are willing to accept arguments with an element of intuition in them, when backed by appeal to a variety of simple and complicated special cases. All the proofs of elementary geometry carry conviction in this way; every property of the triangle is proved for a specific triangle drawn on a sheet of paper or imagined, and there is no guarantee (short of a plunge into the rigor of Hilbert or Veblen and Young) that with a different diagram the theorem may not prove false.

3. The definitions of network theory. The professional electrician is interested in physical networks, consisting of wires, generators, and so on—pieces of apparatus that

actually work. The topologist is interested in undefined elements and axioms concerning them. They both agree that they may play in common a game with pencil marks on a sheet of paper, these marks being to the electrician a representation of his apparatus, and to the topologist a representation of his undefined elements.

In speaking of the marks made on the paper, we may use the language of geometry (points, curves, etc.) or we may (more suitably for present purposes) use the language of the electrician. However, if we take the latter course, we must exercise great care not to read into each physical term its full physical meaning. Thus, to understand the line of thought at a certain point, we must be prepared to use the word "current" without accepting as obvious that the "currents" necessarily obey Kirchhoff's law because all physical currents do. It is to avoid confusions of this sort that we have to be somewhat careful in the matter of definitions.

Let us mark some points on our paper with heavy dots; for reference we may letter them A, B, C, $\cdots$ in any order. These are *nodes* (or terminals or junction-points or vertices), and in general we shall denote their number by N. Figs. 1–4 show cases where $N = 3, 4, 13$ and 34.

Fig. 3. Network with 13 nodes and 25 branches ($N = 13$, $B = 25$).

Next we join the nodes by lines or curves. Every such line or curve has a node at each end. On the lines or curves we put arrows indicating sense, one on each of them, the directions of these arrows being distributed without any plan. The line or curve, with its arrow, is called a *branch* (or directed branch if we want to emphasize that it has a sense). For reference we may attach letters a, b, c, $\cdots$ to the branches, again not according to

any plan, or we may number them in any order; we shall in general denote the number of branches by B. Figs. 1–4 show cases where $B = 3$, 6, 25 and 53.

It is understood that, though two branches may intersect in the diagram, it is forbidden to pass from one branch to another by means of such an intersection. Such a passage can be made only through a node, and we must be careful to distinguish nodes from

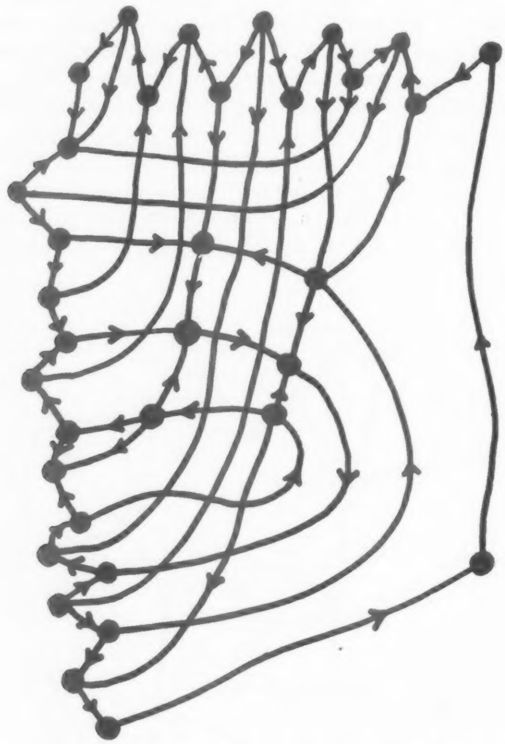

FIG. 4. Network with 34 nodes and 53 branches ($N = 34$, $B = 53$).

intersections by marking the former with heavy dots. This is a trivial nuisance due to the use of a representation on a sheet of paper; it would not arise if we used, as we might, a network of strings in space with knots at the nodes.

Any collection of nodes and branches is a *network*.

The nodes are the ends of branches. In Figs. 1–4 no case is shown where a node is the end of just one branch, although this is allowed by the definition. Such a case appears of little interest from an electrical standpoint, for no current can flow along such a branch. But the idea of a node at the end of a single branch will become important when we discuss trees in the next section.

There is another way in which we have exceeded our instructions in making the networks shown in Figs. 1–4. They are *connected* in the sense that one can travel along branches from any one node to any other. From a topological standpoint, any network

consisting of two (or more) disconnected parts is to be regarded as two (or more) networks. There may be electromagnetic connection between otherwise disconnected parts, but that will not concern us until Section 11 where the argument ceases to be purely topological. For the present we are to think only of connected networks.

We now define a *mesh* (or circuit) by the following prescription. Starting from a node, traverse branches continuously, observing the following rules:

(i) having started to traverse a branch at one end, continue to the other end;

(ii) when you arrive at a node, leave it by a branch other than that by which you arrived, if there is another branch;

(iii) if you arrive at a node which terminates only one branch (that by which you arrived), stop the operation and start all over again.

If in the course of such an operation, you meet the same node (say A) twice, the set of branches you have traversed from A to A form a *mesh*.

To each mesh we assign a sense (one of two possible senses), without regard to the senses already assigned to the branches which compose the mesh. In Fig. 2 *ebda* and *bead* are the same mesh, taken in opposite senses.

4. Trees. A *tree* is a network of a particular type, namely *a connected network containing no meshes.* It is therefore a set of nodes and connecting branches, as shown in Figs. 5, 6, 7.

FIG. 5. Tree with one branch ($N = 2$, $T = 1$; $N = T + 1$).

FIG. 6. Tree with 3 branches ($N = 4$, $T = 3$; $N = T + 1$).

We need certain facts about trees, and these we shall set down as theorems:

THEOREM I: *Every tree has at least one node which is an end of only one branch of the tree.*

This is certainly true of the trees shown, and is easy to prove in general. Simply put your pencil anywhere on a branch of the tree and start moving it along the branches observing the traffic rules laid down in connection with the definition of a mesh. Since there are no meshes in a tree, you cannot get back to the point from which you started. The number of nodes is finite, and so your journey must end. It can end only at a node which is the end of a single branch—the branch by which you have approached it. This proves the theorem.

Since, from any given point on a branch, you have a choice of two directions in which to set out, a tree must contain at least *two* nodes each of which is the end of a single branch. But the theorem as stated is all we need for our purposes.

Theorem II: *In a given connected network it is possible to construct a tree containing all the nodes of the network.*

Figs. 8–11 show trees constructed in the networks of Figs. 1–4, the branches of the trees being shown by heavy wiggly lines in Figs. 8–10 and in a different way in Fig. 11.

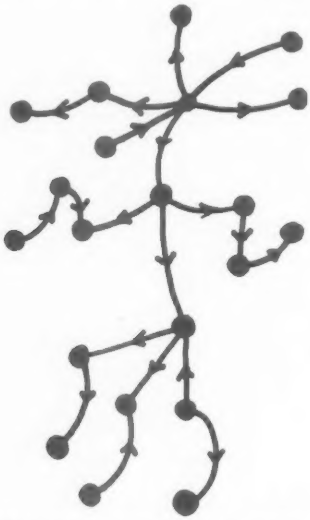

Fig. 7. Tree with 20 branches ($N = 21$, $T = 20$; $N = T + 1$).

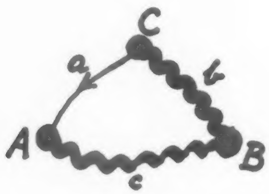

Fig. 8. Tree (heavy wiggly lines) in network of Fig. 1.
2 branches-in-tree; 1 branch-out-of-tree; 1 basic mesh
($N = 3$, $B = 3$, $T = 2$, $M = B - T = 1$; $M + N = B + 1$).

In manuscript a red pencil is best. Further experimentation with networks of his own contriving should convince the reader of the truth of the theorem, but here is a proof.

Start with any branch of the network. It is either a branch of some mesh or it is not. (In Figs. 1–4, it must be a branch of a mesh.) If it is a branch of a mesh, remove the branch; its removal cannot make the network disconnected. If it is not a branch of a mesh, leave it. Testing each branch in this way, and removing it if it belongs to a mesh, we are left at each stage with a connected network. Finally all the meshes will be destroyed, and we shall have a connected network with no meshes, that is, a tree. Since we removed no nodes, the theorem is proved.

This proof makes the construction of a tree seem rather more complicated than it

really is. Set a red pencil on a node, and draw it along a branch to another node. Continue step by step, rejecting beforehand the reddening of any branch which would complete a mesh. A tree will result.

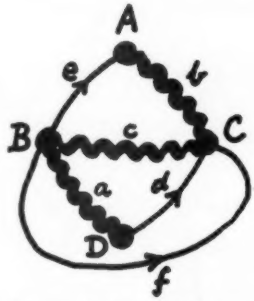

FIG. 9. Tree (heavy wiggly lines) in network of Fig. 2.
3 branches-in-tree; 3 branches-out-of-tree; 3 basic meshes
($N = 4$, $B = 6$, $T = 3$, $M = B - T = 3$; $M + N = B + 1$).

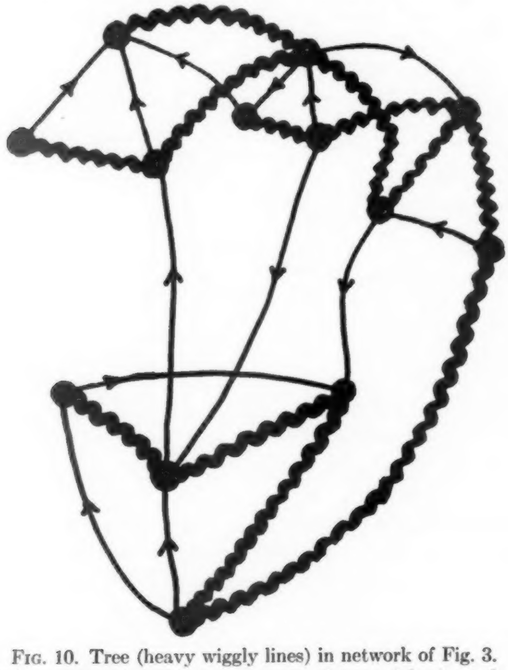

FIG. 10. Tree (heavy wiggly lines) in network of Fig. 3.
12 branches-in-tree; 13 branches-out-of-tree; 13 basic meshes
($N = 13$, $B = 25$, $T = 12$, $M = B - T = 13$; $M + N = B + 1$).

If T is the number of branches of a tree and N the number of nodes, then

$$N = T + 1. \tag{4.1}$$

This is easy to see. For we can build up any tree by starting with a branch and the two

nodes at its ends, and then adding one branch and one node (at the far end of the added branch) at each step. When we start we have $N = 2$, $T = 1$, $N = T + 1$, and since in each step we add unity to T and unity to N, the relation (4.1) holds at any stage, and so for the completed tree.

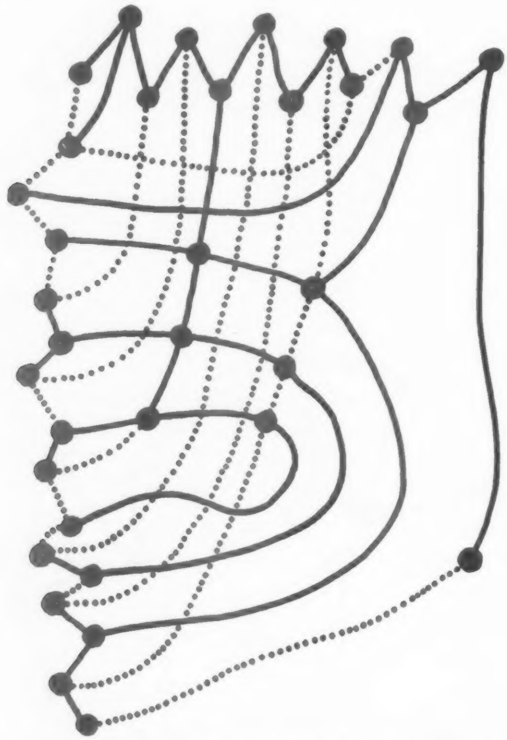

FIG. 11. Tree (tree in full lines, rest of network dotted) in network of Fig. 4. 33 branches-in-tree; 20 branches-out-of-tree; 20 basic meshes ($N = 34$, $B = 53$, $T = 33$, $M = B - T = 20$; $M + N = B + 1$).

If we build up a general connected network like this, we might sometimes add branches without adding nodes. (This does not occur for a tree, for such branches would create meshes, and these a tree, by definition, must not have.) Thus for any connected network we have

$$N \leq B + 1. \tag{4.2}$$

5. Branch currents. So far we have dealt with nodes and branches, and meshes formed of branches.

With a branch we now associate a number i, in general complex. (Note that i is a complex number like $3 + 4j$, where $j = (-1)^{1/2}$.)

We recall that each branch has an arrow on it, pointing from one node (say A) to another node (say B). If i is the current in the branch, we say that a current *i flows into*

the node B, and a current $-i$ *flows into* the node A. (Equivalently we might say that a current $-i$ flows out of B and a current i flows out of A.)

So far the branch currents in a network are to be regarded as a set of B completely arbitrary complex numbers. The whole set may be represented by a matrix $\mathbf{i}$ with B rows and one column.

6. Mesh currents. With a mesh of the network we associate a complex number i', which we call a *mesh current.*

So far branch currents and mesh currents are separate things. We now set up a connection between them, so that we can say that a mesh current *generates* certain branch currents.

Consider any mesh. It contains certain branches, some agreeing in sense with the mesh and some disagreeing. Let us put in a mesh current i'. We say (as a matter of definition) that *this mesh current generates in each branch of the mesh a branch current i' if the senses agree and a branch current $-i'$ if the senses disagree; it generates no branch current in a branch not contained in the mesh.*

Thus in Fig. 2 a mesh current i' in the mesh adc generates branch currents i', i', $-i'$ in a, d, c respectively.

Suppose now that we assign mesh currents in a set of meshes. We say that this set of mesh currents generates a set of branch currents obtained by adding together the branch currents generated by the several mesh currents in accordance with the rule given above.

Thus if in Fig. 2 we put in mesh currents as follows:

$$i_1' \text{ in } ebf,$$

$$i_2' \text{ in } ebc,$$

$$i_3' \text{ in } daeb,$$

these generate the following branch currents:

$$i_a = -i_3',$$

$$i_b = -i_1' - i_2' - i_3',$$

$$i_c = -i_2',$$

$$i_d = -i_3',$$

$$i_e = i_1' + i_2' + i_3',$$

$$i_f = -i_1'.$$

Let us now, in any network, take R meshes and number them $1, 2, \cdots R$. We number the branches $1, 2, \cdots B$. In the meshes we put mesh currents $i_1', i_2', \cdots i_R'$. Then it is clear, from the way in which we have defined the generation of branch currents from mesh currents, that the whole system of branch currents generated by the above mesh currents may be expressed by the formula

$$i_p = \sum_{q=1}^{R} C_{pq} i_q' \qquad (p = 1, 2, \cdots B) \tag{6.1}$$

where the coefficients are numbers defined as follows:

$$C_{pq} \begin{cases} = 1 \text{ if branch } p \text{ is contained in mesh } q \text{ and has the same sense,} \\ = -1 \text{ if branch } p \text{ is contained in mesh } q \text{ and has the opposite sense,} \\ = 0 \text{ if branch } p \text{ is not contained in mesh } q. \end{cases} \tag{6.2}$$

We may write (6.1) in matrix notation:

$$\mathbf{i} = \mathbf{C}\mathbf{i}'. \tag{6.3}$$

Note that the meshes employed here are completely arbitrary.

7. Kirchhoff's node law.* Kirchhoff's node law states that the sum of all branch currents flowing into any node is zero.

If we assign an arbitrary set of branch currents, they will not in general satisfy this law. Indeed, until we examine the question, we cannot be sure that, for a given network, we can find *any* set of branch currents to satisfy the law (except, of course, the trivial set of zero currents).

Consider, however, the branch currents generated by a single mesh current. It is clear that Kirchhoff's node law is satisfied at each node connecting branches of the mesh, and of course at all other nodes $(0 = 0)$. The law remains satisfied if we superimpose any number of mesh currents, and so we have this result:

THEOREM III: *The branch currents generated by any set of mesh currents satisfy Kirchhoff's node law at every node.*

8. Basic meshes. In any given connected network, draw a tree containing all the nodes. All branches of the network then fall into two classes:

(i) Branches-in-tree.

(ii) Branches-out-of-tree (also called *chords*).

The theorem which we wish to prove is this:

THEOREM IV: *Assigning arbitrary branch currents to the branches-out-of-tree, we can assign (and the assignment is unique) branch currents to the branches-in-tree so that Kirchhoff's node law is satisfied at every node.*

We saw in Theorem I that the tree has a node which is an end of only one branch-in-tree. Kirchhoff's node law may be satisfied at that node by assigning a suitable branch current to the branch-in-tree, and of course this current is uniquely determined by the known currents in the branches-out-of-tree.

Now regard the branch-in-tree with which we have been dealing as a branch-out-of-tree, with the branch current we have just found. This leaves us with a reduced tree, and all currents in branches-out-of-tree assigned. Treat this reduced tree in the same way. Step by step, we convert branches-in-tree into branches-out-of tree, each with an assigned current, and Kirchhoff's law is satisfied at every node which we drop from the tree. Ultimately we reach a stage where the tree consists of a single branch.

At each end of this single branch-in-tree, Kirchhoff's law determines a current for it. For a moment it may appear that our method might break down here—the two ends of

*As there is a possible confusion (Ingram and Cramlet, *loc. cit.*) as to which should be called Kirchhoff's "first" law and which his "second" law, the unambiguous words "node" and "mesh" will be used here.

the branch might demand different currents. But that cannot occur. When we started, the total current flowing into the tree at all the nodes was zero, since each branch-out-of-tree made equal and opposite contributions at its two ends. This zero total flow is preserved each time we change a branch-in-tree into a branch-out-of-tree, and so when we have reduced the tree to a single branch, the flows into its two ends are equal and opposite in sign. Thus we can get rid of the last branch of the tree, turning it into a branch-out-of-tree with a uniquely assigned current. Then Kirchhoff's node law is satisfied at every node and the theorem is proved.

For any connected network we now define *basic meshes*, using a tree in the network, containing all the nodes. This is very simple. Each branch-out-of-tree has its ends connected by a unique set of branches-in-tree (necessarily unique, since otherwise the tree would contain a mesh), and together these form a mesh; to it we assign the sense of the branch-out-of-tree contained in it. Thus, if there are M branches-out-of-tree, there are M meshes defined in this way, each mesh containing only one branch-out-of-tree, and each branch-out-of-tree being contained in just one mesh. We call this a set of *basic meshes* of the given network. Once the tree has been chosen, the basic meshes are uniquely determined; but since in general there will be several trees which may be used, so there are several sets of basic meshes.

The total number of nodes in the network being N, the number of branches-in-tree is, by (4.1), $T = N - 1$. The total number of branches in the network being B, the number of branches-out-of-tree (the same as the number of basic meshes) is thus

$$M = B - T = B - N + 1, \tag{8.1}$$

or equivalently

$$M + N = B + 1. \tag{8.2}$$

This general topological relation for a connected network is a useful check in the case of complicated networks as in Figs. 10 and 11.

Although topology (and, more generally, modern geometry) is a severe logical discipline, it seems right to use the word "topology" as we use the word "geometry" to describe properties of systems of dots and curves drawn on paper, or a net of knotted strings, or even an electrical network. In speaking of an electrical network, we try to clear our thoughts by separating its "topological" properties from its "electrical" properties. As long as the properties of which we speak are properties shared by systems of dots and curves and knotted strings, then we may say that we are in the topological domain. It is when we start connecting emfs and currents that we pass into electrical theory proper.

In this sense, then, almost all that has been done up to this point, and indeed almost all until we reach Section 11, is topology. The mesh and the tree belong to topology, and so do the basic meshes, since they are derived directly from the tree. This is not a matter for controversy regarding rival spheres of influence. It is an attempt to assist understanding by analysing and classifying the elements which, taken all together and without analysis and classification, can be bewilderingly confused.

9. Kron's transformation matrix. Suppose we have a connected network before us. We pick out a tree, as we always can, and hence a set of basic meshes. Let there be M basic meshes. We number the branches $1, 2, \cdots B$ and the basic meshes $1, 2, \cdots M$, with no attention to any particular order.

To the basic meshes we assign arbitrary mesh currents $i'_1, i'_2, \cdots i'_M$. Now (6.1) applies

for any meshes, and so we can express the branch currents generated by the currents in the basic meshes by the formula

$$i_p = \sum_{q=1}^{M} C_{pq} i'_q , \qquad (p = 1, 2, \cdots B) \tag{9.1}$$

where

$$C_{pq} \begin{cases} = 1 \text{ if branch } p \text{ is contained in mesh } q \text{ and has the same sense,} \\ = -1 \text{ if branch } p \text{ is contained in mesh } q \text{ and has the opposite sense,} \\ = 0 \text{ if branch } p \text{ is not contained in mesh } q. \end{cases} \tag{9.2}$$

In matrix form we have

$$\mathbf{i} = \mathbf{C}\mathbf{i}'; \tag{9.3}$$

here the $B \times M$ matrix $\mathbf{C}$ is Kron's *transformation matrix*.

We note that, by Theorem III, if the mesh currents are chosen arbitrarily, the branch currents given by (9.1) or (9.3) necessarily satisfy Kirchhoff's node law at every node.

But there is another important thing which we have to prove, namely, that by proper choice of the mesh currents $\mathbf{i}'$ (9.1) or (9.3) will yield the most general set of branch currents $\mathbf{i}$ consistent with Kirchhoff's node law.

To prove this, let us assign any set of branch currents consistent with Kirchhoff's node law. Then consider the tree in the network which corresponds to the basic meshes involved, and think of the branches-in-tree and the branches-out-of-tree.

Each basic mesh contains just one branch-out-of-tree, and the branch current in it has been assigned. To the mesh containing that branch assign a mesh current equal to that branch current. Having done this for all the basic meshes, we have a set of mesh currents which generates a set of branch currents. We have deliberately made this set of branch currents coincide with the assigned branch currents in the branches-out-of-tree, and the coincidence for the branches-in-tree (and hence for all branches) follows from Theorem IV. Thus the assigned branch currents, arbitrary except for the satisfaction of Kirchhoff's node law, can be generated by a suitably chosen set of mesh currents in the basic meshes.

Let us summarise this result. It is the central point; what has gone before was preparation for it, and what follows is easy deduction. It is the central theorem involved in Kron's formula (9.3), or indeed in any mesh-branch transition in network theory. This is the theorem:

THEOREM V: *Given a network, there exists at least one set of basic meshes such that the following statements are true, the matrix* $\mathbf{C}$ *being defined as in (9.2):*

(i) *If a set of mesh currents* $\mathbf{i}'$ *is arbitrarily assigned, the branch currents given by* $\mathbf{i} = \mathbf{C}\mathbf{i}'$ *satisfied Kirchhoff's node law at every node.*

(ii) *If a set of branch currents* $\mathbf{i}$ *is assigned, arbitrarily except for the satisfaction of Kirchhoff's node law at every node, there exists a set of mesh currents* $\mathbf{i}'$ *in the basic meshes such that* $\mathbf{i} = \mathbf{C}\mathbf{i}'$.

10. Branch emfs and mesh emfs. For present purposes a *branch emf* e is simply a complex number associated with a branch of a network, and a *mesh emf* e' is a complex number associated with a mesh.

We bring these two concepts into relation with one another in a manner complementary to that in which we brought together branch currents and mesh currents. We say

that if branch emfs are assigned in all the branches, *they generate in any mesh a mesh emf equal to the sum of the branch emfs in the branches contained in the mesh, a + or − sign being prefixed to the branch emf before adding according as the senses of the branch and the mesh agree or disagree.*

If, as usual, we number the branches 1, 2, $\cdots$ B and the meshes 1, 2, $\cdots$ M (we take a basic set, although the immediate statement holds more generally), then the above verbal statement is obviously equivalent to the following formula conneeting mesh emfs e' and branch emfs e:

$$e'_q = \sum_{p=1}^{B} D_{qp} e_p \qquad (q = 1, 2, \cdots M), \tag{10.1}$$

where the coefficients are defined by

$$D_{qp} \begin{cases} = 1 \text{ if branch } p \text{ is contained in mesh } q \text{ and has the same sense,} \\ = -1 \text{ if branch } p \text{ is contained in mesh } q \text{ and has the opposite sense,} \\ = 0 \text{ if branch } p \text{ is not contained in mesh } q. \end{cases} \tag{10.2}$$

When we compare this with (9.2), we see at once that $D_{qp} = C_{pq}$, or in matrix language $\mathbf{D} = \mathbf{C}_t$ where $\mathbf{C}_t$ is the transpose of $\mathbf{C}$. Thus (10.1) may be written

$$e'_q = \sum_{p=1}^{B} C_{pq} e_p , \tag{10.3}$$

or

$$\mathbf{e}' = \mathbf{C}_t \, \mathbf{e} \cdot \tag{10.4}$$

To sum up:

THEOREM VI: *Given a network and a basic set of meshes in it, any assigned set of branch emfs* $\mathbf{e}$ *generate mesh emfs* $\mathbf{e}'$ *according to the formula (10.4), where* $\mathbf{C}_t$ *is the transpose of Kron's transformation matrix.*

We have then at this stage obtained, with it is hoped a proper understanding of their meanings, the two formulae

$$\mathbf{i} = \mathbf{C}\mathbf{i}', \qquad \mathbf{e}' = \mathbf{C}_t\mathbf{e}. \tag{10.5}$$

The currents and emfs have appeared so far only as numbers associated with branches and meshes, without electrical significance except in connection with Kirchhoff's node law and the generation rules connecting quantities for branches and meshes. There has been no reference at all to an impedance matrix.

We have been thinking of a single connected network. If we have a set of disconnected networks, the formulae (10.5) hold for each of them, and it is easy to see (such is the elasticity of matrix notation) that the whole set of such relations, a pair for each connected network, may be compressed into a pair of formulae of precisely the form (10.5), now covering the whole network.

This ends what may be called the purely topological part of the paper.

11. How to solve a network. Suppose we have a network, in general consisting of several parts, disconnected topologically but connected electromagnetically. Let the branches (B in number) contain generators of emf, d.c. or a.c. A square $B \times B$ impedance

matrix $\mathbf{Z}$ is given. The problem of solving the network consists in finding the branch currents i, given the branch emfs e, only steady states being considered.*

We cannot of course attempt to do this without some law connecting currents and emfs. To state this law, we define a set of B *potential-differences* $\mathbf{W}$ in the branches by the formula

$$\mathbf{W} = \mathbf{e} - \mathbf{Zi}. \tag{11.1}$$

We then accept Kirchhoff's mesh law, which asserts that *the sum of the potential differences for the branches of any mesh is zero, each potential-difference being prefixed before adding with a + or − sign according as the senses of the branch and the mesh agree or disagree.*

This holds for *any* mesh, and we could continue our discussion for a while using any meshes at all. But to avoid unnecessary generality, let us apply Kirchhoff's mesh law to a set of basic meshes. It is clear that the verbal statement made above is precisely equivalent to

$$\sum_{p=1}^{B} D_{qp} W_p = 0, \tag{11.2}$$

where D_{qp} is defined by (10.2). Thus Kirchhoff's mesh law gives

$$\mathbf{C}_t \mathbf{W} = 0, \tag{11.3}$$

since, as we saw, $\mathbf{D}$ is the transpose of $\mathbf{C}$.

We now do some formal work with matrices. Multiplication of (11.1) on the left by $\mathbf{C}_t$ gives

$$\mathbf{C}_t \mathbf{e} - \mathbf{C}_t \mathbf{Z i} = 0, \tag{11.4}$$

and, by (10.5), this may be written

$$\mathbf{e}' = \mathbf{Z}'\mathbf{i}', \tag{11.5}$$

where $\mathbf{Z}'$ is defined to be

$$\mathbf{Z}' = \mathbf{C}_t \mathbf{Z C}. \tag{11.6}$$

We have not achieved our objective; we have not found the branch currents $\mathbf{i}$ in terms of the branch emfs $\mathbf{e}$. We have got no further than (11.5) which expresses mesh emfs in terms of mesh currents. And we can go no further without an additional assumption which will assure us that the matrix $\mathbf{Z}'$ is not singular.

12. The final step. The final assumption which we might make is that the network is *dissipative*, i.e. it contains resistances. We know then that if all emfs are put equal to zero, there cannot exist steady d.c. or a.c. currents, except of course zero currents. In the case of a.c. we can be more general, and say that the network may be resistanceless, but that the frequencies which we consider do not include resonant frequencies of the network; then in the absence of all emfs there can exist no periodic currents in the network, except of course the trivial zero currents.

The essential assumption is that the network (and perhaps the frequencies considered) should be such that $\mathbf{e} = 0$ *implies* $\mathbf{i} = 0$. This is the assumption we shall make.

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* In the case of *a. c.*, i and e are complex constants from which the physical quantities are obtained by multiplying by exp (jpt) and then taking the real part.

Consider now the equation (11.5). The matrix $\mathbf{Z}'$, as given by (11.6), is in part topological (through $\mathbf{C}$) and in part electromagnetic (through $\mathbf{Z}$); but it is defined by the *structure* of the network, and is independent of the values of the branch emfs.

Let us then put $\mathbf{e} = 0$. This implies, under our assumption, $\mathbf{i} = 0$. Going back to the tree from which the basic meshes were constructed in Section 8, we see that each mesh current is actually a branch current, namely, in a branch-out-of-tree. Hence $\mathbf{i} = 0$ implies $\mathbf{i}' = 0$. Now $\mathbf{e} = 0$ certainly implies $\mathbf{e}' = 0$ by (10.5). We conclude from (11.5) that the equations

$$\mathbf{Z}'\mathbf{i}' = 0 \tag{12.1}$$

are satisfied only by $\mathbf{i}' = 0$.

This means that the determinant of the matrix $\mathbf{Z}'$ is not zero; the matrix is therefore non-singular, and $\mathbf{Z}'^{-1}$ exists.

The rest is simple. Restoring general values to $\mathbf{e}$, we multiply (11.5) on the left by $\mathbf{Z}'^{-1}$ and get

$$\mathbf{i}' = \mathbf{Z}'^{-1}\mathbf{e}'. \tag{12.2}$$

Then we multiply on the left by $\mathbf{C}$ and use both the equations (10.5): this gives

$$\mathbf{i} = \mathbf{C}\mathbf{Z}'^{-1}\mathbf{C}_t\mathbf{e}. \tag{12.3}$$

or, inserting the value of $\mathbf{Z}'$ from (11.6),

$$\mathbf{i} = \mathbf{C}(\mathbf{C}_t\mathbf{Z}\mathbf{C})^{-1}\mathbf{C}_t\mathbf{e}, \tag{12.4}$$

which is Kron's fundamental equation for the solution of networks.

13. Conclusion. The essential steps leading ultimately to (12.4) are as follows:

(a) Deduction of $\mathbf{i} = \mathbf{C}\mathbf{i}'$; this is topology, oriented by Kirchhoff's node law.

(b) Deduction of $\mathbf{e}' = \mathbf{C}_t\mathbf{e}$; very simple when we have done (a).

(c) Deduction of $\mathbf{e}' = \mathbf{Z}'\mathbf{i}'$ from Kirchhoff's mesh law.

(d) An assumption assuring the non-singular character of $\mathbf{Z}'$, so that we can solve for $\mathbf{i}$.

STUDIES ON TWO-DIMENSIONAL TRANSONIC FLOWS OF COMPRESSIBLE FLUID.—PART III*

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13. The introduction of a second new hypothetical gas. In the foregoing sections, we have investigated some transonic fields of flow by employing a hypothetical gas which closely approximates the real gas during an isentropic flow in the transonic range. Although such a hypothetical gas has a merit that the fundamental equation governing its flow assumes a rather simple form and can be solved exactly in several cases, it has the drawback that it can approximate the real gas obeying the isentropic law only for a limited (transonic) range of velocities. Thus, the method of analysis as developed in the foregoing sections has a rather narrow application; it can be applied only to nearly uniform transonic flows.

In Part III, an attempt is made to develop a theory which is applicable even to a transonic flow containing limited supersonic regions as well as stagnation points. For this purpose, we have introduced a second hypothetical gas which is capable of representing the real gas subject to the isentropic law with a better degree of approximation than that used in Parts I and II.

For the sake of convenience, we start from the linearized equations of motion in the hodograph plane which are valid for any compressible perfect fluid. They can be written in the following forms:

$$\begin{aligned} \varphi_w &= -X\psi_\theta , \\ \varphi_\theta &= \psi_w , \end{aligned} \tag{13.1}$$

where, as before, the coefficient X and independent variable w are respectively given by

$$\begin{aligned} X &= -\frac{q^2}{\rho}\left(\frac{1}{\rho q}\right)' = \frac{q^2}{\rho^2}\left(\frac{1}{q^2} - \frac{1}{c^2}\right), \\ w &= \int_1^q \frac{\rho}{q}\, dq. \end{aligned} \tag{13.2}$$

Eliminating φ from Eqs. (13.1), we obtain the fundamental equation for determining ψ in the form:

$$\psi_{ww} + X\psi_{\theta\theta} = 0. \tag{13.3}$$

The coefficient X in the second term can evidently be expressed as a function of w alone by the use of the known equation of state of any gas together with Bernoulli's theorem. The dotted-line curve in Fig. 20 shows the behaviour of $X(w)$ for the case of the real gas obeying the isentropic law, the value of γ having been taken equal to 1.4 for air.

It is in general difficult to obtain exact solutions of the fundamental equation (13.3)

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with the coefficient $X(w)$ for the case of the real gas. It is therefore suggested that the function $X(w)$ for the case of the real gas may be replaced by a simple approximate function in order to reduce Eq. (13.3) to a tractable form. The most simple replacement of $X(w)$ by a suitable constant (as done by Chaplygin, Kármán and Tsien), however, is not appropriate for the investigation of transonic flow. In Parts I and II, we have replaced the function $X(w)$ for the case of the real gas by the tangent, as shown by the

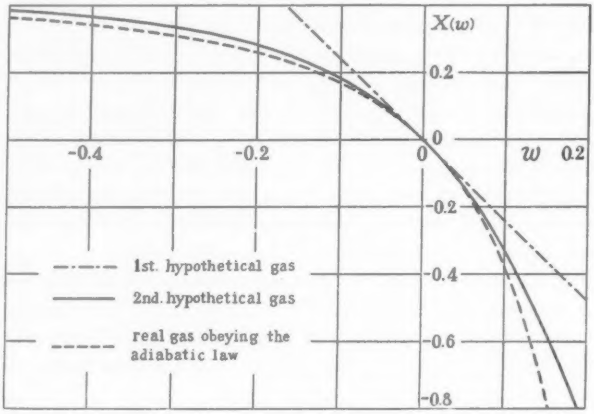

FIG. 20.

chain-line curve in Fig. 20, at the point $w = 0$ (i.e. at the critical state $q = c = 1$) for the purpose of dealing with nearly uniform transonic flow. In order to treat transonic fields of flow containing stagnation points, however, it is necessary to make use of another approximate function which can approximate more precisely the function $X(w)$ for the real gas.

In the following lines, we shall use a function of the form:

$$X(w) = a(1 - be^{2\kappa w}), \tag{13.4}$$

where a, b and κ are constants to be determined adequately. As will be seen presently, the use of this function makes our fundamental equation (13.3) tractable.

We have conveniently determined the values of the constants a, b and κ in (13.4) in such a way that the curve of $X(w)$ as given by (13.4) coincides with the corresponding curve for the real gas to the order of their tangents at $w = -\infty$ (which corresponds to the stagnation point) as well as at $w = 0$ (which corresponds to the critical state). The expression for $X(w)$ thus determined is given by

$$X(w) = a(1 - e^{2\kappa w}),$$

with (13.5)

$$a = \left(\frac{2}{\gamma + 1}\right)^{2/(\gamma-1)}, \qquad \kappa = \left(\frac{\gamma + 1}{2}\right)^{(\gamma+1)/(\gamma-1)}.$$

The full-line curve in Fig. 20 shows the curve of $X(w)$ given by (13.5), by taking, as before, $\gamma = 1.4$ for air. It will readily be observed that this curve can satisfactorily ap-

proximate the curve of $X(w)$ for the real gas which is shown by the dotted curve in the figure.

Now, as mentioned already, the form of the function $X(q)$ defined by (13.2) depends upon the equation of state $\rho(q)$ of the gas concerned. Conversely, any given expression of this function $X(q)$ determines the equation of state of a corresponding gas. Thus, by introducing, in the following analysis, a second new hypothetical gas, we take the function defined by (13.5) as an exact relation valid for such a hypothetical gas, instead of considering it as an approximation to the corresponding function for the real gas obeying the isentropic law, and we shall deal, in an exact manner, with the field of flow

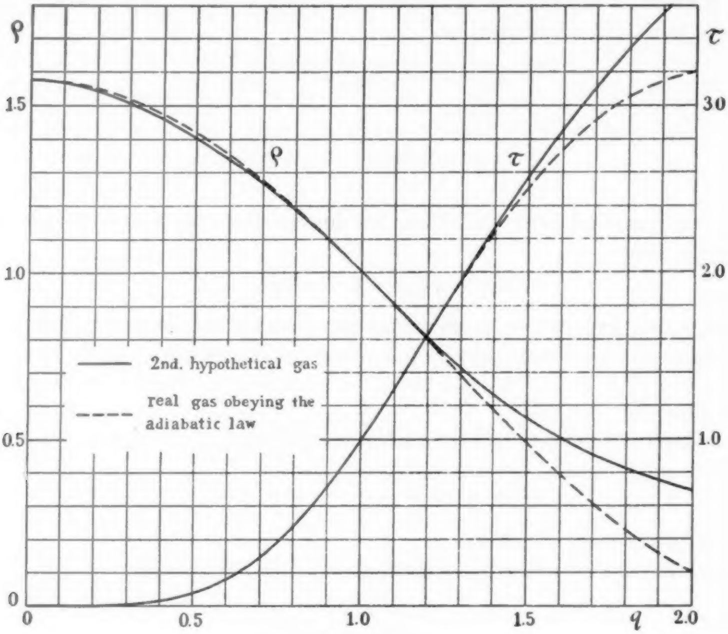

FIG. 21.

of such a hypothetical gas, as done with the previous hypothetical gas in Parts I and II.

If we insert the expression for $X(w)$ given by (13.5) into Eq. (13.3), we obtain the fundamental equation for determining the flow of our new hypothetical gas in the form.

$$\psi_{ww} + a(1 - e^{2\kappa w})\psi_{\theta\theta} = 0,$$

with

$$a = \left(\frac{2}{\gamma + 1}\right)^{2/(\gamma-1)}, \qquad \kappa = \left(\frac{\gamma + 1}{2}\right)^{(\gamma+1)/(\gamma-1)}. \tag{13.6}$$

Before proceeding further, we shall now discuss briefly the properties of our hypothetical gas. If we combine (13.2) with (13.5), we obtain the differential equation for determining the equation of state $\rho(q)$ of our new hypothetical gas. We thus have

$$2\kappa \frac{\rho}{q}\left\{a + \frac{q^2}{\rho}\left(\frac{1}{\rho q}\right)'\right\} = \left\{\frac{q^2}{\rho}\left(\frac{1}{\rho q}\right)'\right\}'. \tag{13.7}$$

Solving this equation, by the method of numerical integration or otherwise, under the conditions that $\rho = 1$ and $\rho' = -1$ at $q = 1$, we have obtained the curve of $\rho(q)$ for our hypothetical gas as shown by a full-line curve in Fig. 21. It will be seen that this curve of $\rho(q)$ coincides with the corresponding curve, shown by a dotted-line curve, for the real gas obeying the isentropic law, to the order of tangent at $q = 0$ and to the order of curvature at $q = 1$. In this figure, the curves of $\tau(q)$ for both our hypothetical gas and the real gas subject to the isentropic law are also given by a full-line curve and a dotted-line curve respectively, where $\tau = e^{\kappa w}$ is a new variable to be introduced in the following section.

14. A method of solving the fundamental equation (13.6). We now introduce new independent variables τ, β defined as:

$$\tau = e^{\kappa w} = \exp\left(\kappa \int_1^q \frac{\rho}{q}\, dq\right), \qquad \beta = \frac{\kappa}{\sqrt{a}}\,\theta. \tag{14.1}$$

Then, the fundamental equation (13.6) for determining the flow of our hypothetical gas takes the form:

$$\tau^2 \psi_{\tau\tau} + \tau\psi_\tau + (1 - \tau^2)\psi_{\beta\beta} = 0. \tag{14.2}$$

It is evident that just as the fundamental equation for the isentropic flow of the real gas does, this equation (14.2) changes from the elliptic to the hyperbolic type according as $\tau < 1$ (i.e. $q < 1$) or $\tau > 1$ (i.e. $q > 1$), i.e. according to whether the flow is subsonic or supersonic.

The characteristic curves of Eq. (14.2) are given by the equations:

$$\pm(\beta - \beta_0) = \sqrt{\tau^2 - 1} - \cos^{-1}\frac{1}{\tau}, \tag{14.3}$$

where β_0 is an arbitrary parameter.

To solve Eq. (14.2), we first assume that

$$\psi = T(\tau)e^{-in\beta}, \tag{14.4}$$

with an arbitrary constant n. Then, we obtain an ordinary differential equation for determining the function $T(\tau)$ in the form:

$$\tau^2 \frac{d^2T}{d\tau^2} + \tau \frac{dT}{d\tau} - n^2(1 - \tau^2)T = 0. \tag{14.5}$$

The general solution of this equation can be expressed in terms of Bessel functions. Thus,

$$T(\tau) = AJ_n(n\tau) + BY_n(n\tau),$$

where A and B are arbitrary constants. Hence, the solution of (14.2) which is finite at $\tau = 0$ can be expressed in the form:

$$\psi = \sum_{n=0}^{\infty} A_n J_n(n\tau)e^{-in\beta}, \tag{14.6}$$

where n and A_n's are arbitrary constants.

If now we make use of the integral representation of $J_n(n\tau)$ of the Bessel type, namely[1]:

$$J_n(n\tau) = \frac{1}{2\pi i} \oint_{(0+)} \{t^{-1} \exp [\tau(t - 1/t)/2]\}^n \frac{dt}{t},$$

the expression for ψ becomes

$$\psi = \frac{1}{2\pi i} \oint_{(0+)} \sum_{n=0}^{\infty} A_n\{t^{-1} \exp [\tau(t - 1/t)/2 - i\beta]\}^n \frac{dt}{t}. \tag{14.7}$$

Since, however, the series in the integrand:

$$\sum_{n=0}^{\infty} A_n\{t^{-1} \exp [\tau(t - 1/t)/2 - i\beta]\}^n$$

is evidently the expansion of a certain function in powers of $t^{-1} \exp [\tau(t - 1/t)/2 - i\beta]$, the above solution can be generalized in the following form:

$$\psi = \frac{1}{2\pi i} \int_C F(Z) \frac{dt}{t},$$

with (14.8)

$$Z = t^{-1} \exp [\tau(t - 1/t)/2 - i\beta],$$

where $F(Z)$ denotes an arbitrary function of the variable Z, and the path of integration C should be so chosen that this expression for ψ may really become the solution of Eq. (14.2).

In fact, inserting the above expression (14.8) for ψ into the left-hand side of Eq. (14.2), which is conveniently denoted by $D(\psi)$, we have

$$D(\psi) = \frac{1}{2\pi i} \int_C \frac{d}{dt} \left[\left\{ \frac{\tau}{2} \left(t + \frac{1}{t} \right) + 1 \right\} Z \frac{dF}{dZ} \right] dt.$$

Thus, it will be seen that in order that the function ψ given by (14.8) becomes in effect the solution of Eq. (14.2), i.e. $D(\psi) = 0$, the function:

$$\Delta \equiv \left\{ \frac{\tau}{2} \left(t + \frac{1}{t} \right) + 1 \right\} Z \frac{dF}{dZ}, \qquad (Z = t^{-1} \exp [\tau(t - 1/t)/2 - i\beta])$$

must take the same value at the two end-points of the path of integration C and that in the case where the path C is a closed curve, it is sufficient that dF/dZ should be one-valued along C.

15. A few fundamental solutions of equation (14.2). In this section, a few fundamental solutions will be constructed by applying the method explained above.

In the first place, we shall consider a solution which is obtained by taking the arbitrary constant A_n in the general solution (14.6) to be equal to $1/\lambda^n$, where λ is a certain positive constant. By comparison with the results obtained in §§8, 9 and 10 of Part II, it is expected that such a solution will have a branch-point of order $-1/2$ at a certain point in the hodograph plane, the position of the singular point depending however upon the value of the parameter λ.

[1] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press, 1922, p. 20.

Since $A_n = 1/\lambda^n$, the function F in (14.8) becomes

$$F = \frac{1}{1 - e^z}$$

with (15.1)

$$z = \frac{\tau}{2}\left(t - \frac{1}{t}\right) - \log t - i\beta - \log \lambda.$$

Thus, if we denote the solution under consideration by $\psi_{(-1/2)}$, we have

$$\psi_{(-1/2)} = \frac{1}{2\pi i}\int_C \frac{dt}{t(1 - e^z)}, \tag{15.2}$$

where C is a path of integration to be so chosen that this $\psi_{(-1/2)}$ actually becomes the solution of the fundamental equation (14.2).

Let $t = t_0$ be a root of the equation $z(t) = 0$.[2] Then, in the neighbourhood of the point $t = t_0$, the integrand in (15.2) can be expanded in the following form:

$$\frac{1}{t(1 - e^z)} = -\frac{1}{t_0(dz/dt)_{t=t_0}}\frac{1}{t - t_0} + O(1),$$

and therefore it will be seen that the integrand has a pole of the first order at the point $t = t_0$, provided that $(dz/dt)_{t=t_0} \neq 0$.

It is readily found that if we take, as the path of integration C in (15.2), a small closed curve enclosing only the point $t = t_0$ but not any other singular points, the function $\psi_{(-1/2)}$ given by (15.2) actually becomes the solution of the fundamental equation (14.2). The value of the integral can then be obtained by evaluating the residue of the integrand at the point $t = t_0$ and thus, taking the integral along the path of integration in the positive sense, we have

$$\psi_{(-1/2)} = -\frac{1}{t_0(dz/dt)_{t=t_0}} = \frac{1}{1 - (\tau/2)(t_0 + 1/t_0)}.$$

If we here put $t_0 = e^{i\omega}$ for the sake of convenience, the expression for $\psi_{(-1/2)}$ becomes ultimately

$$\psi_{(-1/2)} = \frac{1}{1 - \tau\cos\omega}. \tag{15.3^3}$$

On the other hand, since $t = t_0$ has been assumed to be a zero-point of the function $z(t)$ as defined by (15.1), we have $z(t_0) = 0$, and when use is made of the above substitution $t_0 = e^{i\omega}$, this equation becomes

$$\beta - i\log\lambda + \tau\sin\omega - \omega = 0. \tag{15.4}$$

By eliminating ω from the above two equations (15.3) and (15.4), we can express $\psi_{(-1/2)}$ in terms of τ and β.

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[2] It may be remarked here that this equation has in general two different roots.

[3] It can be ascertained without difficulty that this $\psi_{(-1/2)}$ really becomes a solution of the fundamental equation (14.2).

For a special set of values τ_∞, β_∞ of the independent variables τ, β, the two equations:

$$\begin{aligned} \beta - i \log \lambda + \tau \sin \omega - \omega &= 0, \\ 1 - \tau \cos \omega &= 0 \end{aligned} \tag{15.5}$$

are satisfied simultaneously,[4] and the function $\psi_{(-1/2)}$ as given by (15.3) becomes infinite. Thus, it is found that as has been expected from the outset, the solution $\psi_{(-1/2)}$ has a branch-point of order $-1/2$ at the point $(\tau_\infty, \beta_\infty)$ in the τ, β plane.

Elimination of ω from the two equations in (15.5) gives the equation for determining τ_∞ and β_∞ in the form:

$$\beta_\infty + i\left\{-\log \lambda \pm \sqrt{1 - \tau_\infty^2} \mp \cosh^{-1} \frac{1}{\tau_\infty}\right\} = 0. \tag{15.6}$$

In case when $\tau_\infty \leqq 1$, both $\sqrt{1 - \tau_\infty^2}$ and $\cosh^{-1}(1/\tau_\infty)$ are real and therefore, separating the real and imaginary parts, we have

$$\begin{aligned} \beta_\infty &= 0, \\ -\log \lambda \pm \sqrt{1 - \tau_\infty^2} \mp \cosh^{-1} \frac{1}{\tau_\infty} &= 0. \end{aligned} \tag{15.7}$$

From these equations it will be seen that the singular point of the solution $\psi_{(-1/2)}$ in the τ, β plane is, in this case, situated at an isolated point $\tau = \tau_\infty$ on the axis $\beta = 0$. The second equations in (15.7) give the relationship between the coordinate τ_∞ and the parameter λ. The curve of τ_∞ plotted against λ is shown in Fig. 22, from which it is

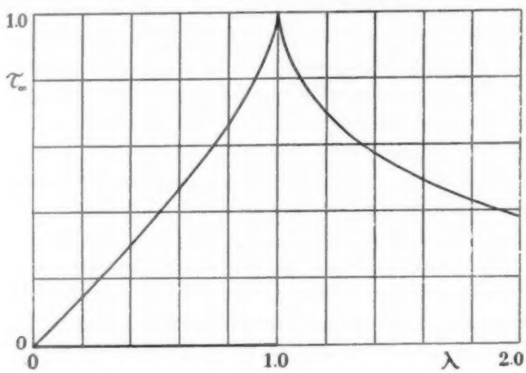

FIG. 22.

readily found that in order to make τ_∞ to have any value in the range from 0 to 1, it is quite sufficient to take for λ only a value in the restricted range from 0 to 1.

Further, for the hypothetical gas under consideration, we have, from (13.2), (13.5) and (14.1),

$$\frac{q^2}{\rho^2}\left(\frac{1}{c^2} - \frac{1}{q^2}\right) = a(\tau^2 - 1).$$

[4]It is readily found that this corresponds to the case when $(z)_{t=t_0} = 0$ and $(dz/dt)_{t=t_0} = 0$, i.e. when the two roots of the equation $z(t) = 0$ coincide with each other.

Therefore, the Mach number M associated with the state at the singular point $\tau = \tau_\infty$, $\beta = 0$ is calculated by the formula:

$$M = \frac{q_\infty}{c_\infty} = \sqrt{1 + a\rho_\infty^2(\tau_\infty^2 - 1)}. \tag{15.8}$$

Next, we shall consider the case when $\tau_\infty \geqq 1$. In this case, both $\sqrt{1 - \tau_\infty^2}$ and $\cosh^{-1}(1/\tau_\infty)$ are purely imaginary and therefore, by separating the real and imaginary parts on the right-hand side of Eq. (15.6), we have the following two equations:

$$\begin{aligned} \log \lambda &= 0, \\ \beta_\infty \mp \sqrt{\tau_\infty^2 - 1} \pm \cos^{-1}\frac{1}{\tau_\infty} &= 0. \end{aligned} \tag{15.9}$$

Hence, it will be seen that in the supersonic region where $\tau \geqq 1$, the singularity occurs only at a particular value $\lambda_{max} = 1$ of the parameter λ, and this time the solution $\psi_{(-1/2)}$ becomes infinite, not at an isolated point but along two curves in the τ, β plane which satisfy the second equations in (15.9).

Further, by comparing (15.9) with (14.3), it is found that these curves of singularity are nothing less than the two characteristic curves passing through the point $\tau = 1$, $\beta = 0$.

Thus, summarizing the above results we see that when the parameter λ assumes a value in the range $0 < \lambda < 1$, the singularity of the solution $\psi_{(-1/2)}$ in the hodograph plane (i.e., the τ, β plane) occurs at an isolated point on the axis $\beta = 0$, but in the limit $\lambda \rightarrow 1$, the singularity is prolonged along the two characteristics passing through the point $\tau = 1$, $\beta = 0$. This remarkable characteristic change of the singularity of the solution occurring in the hodograph plane at the stage of transition from the subsonic to the supersonic region is quite similar to what has been already found in Part II in the case of the more simple fundamental equation (7.6) of the mixed type.

Lastly, we shall derive from the preceding solution $\psi_{(-1/2)}$ another solution which will have a branch-point of order 1/2 at a certain point in the hodograph plane.

Now, in general, the form of our fundamental equation (14.2) suggests that any new solution can be obtained by differentiating or integrating one of the known solutions with respect to β, and as inferred from what has been pointed out in Part II, the order of singularity of the solution thus derived would differ by unity from the order of the original solution. Hence, the required solution, denoted by $\psi_{(1/2)}$, can be derived from the preceding solution $\psi_{(-1/2)}$ given by (15.3) by integrating it with respect to β, and we have

$$\psi_{(1/2)} = \int \frac{d\beta}{1 - \tau \cos \omega} = \int \frac{1}{1 - \tau \cos \omega} \frac{d\omega}{d\omega/d\beta} = \int d\omega = \omega,$$

where use has been made of the relation $d\omega/d\beta = (1 - \tau \cos \omega)^{-1}$ which can easily be obtained from (15.4). It is thus found that the variable ω itself is a solution having the singularity of order 1/2.

On the other hand, the variable β itself also becomes evidently a solution of the fundamental equation (14.2). Therefore, it will be seen from equation (15.4) that the function $\tau \sin \omega$ becomes also a solution having the singularity of order 1/2. Thus, denoting this solution by $\psi_{(1/2)}$ as before, we have

$$\psi_{(1/2)} = \tau \sin \omega. \tag{15.10}$$

It can easily be proved that the singularity of these solutions $\psi_{(1/2)}$ show also the characteristic change as mentioned above, just as in the case of the preceding solution $\psi_{(-1/2)}$.

As will be shown in later lines, we can discuss a uniform flow past an obstacle by making use of an appropriate linear combination of the above solutions $\psi_{(-1/2)}$ and $\psi_{(1/2)}$.

16. An alternative method of solving the fundamental equation (14.2). In the analysis developed in §14, we have used the integral representation of Bessel's type for the Bessel function $J_n(n\tau)$ in (14.6). However, if, instead of employing the integral representation of Bessel's type, we make use of the integral representation of Poisson's type for $J_n(n\tau)$, we can develop an alternative analysis similar to the preceding one, which enables us to calculate conveniently the limiting case of the incompressible fluid flow.

Now, according to Poisson, the Bessel function $J_n(n\tau)$ can be expressed in the form:[5]

$$J_n(n\tau) = \frac{(n\tau/2)^n}{\Gamma(n + 1/2)\Gamma(1/2)} \int_0^\pi \exp\{in\tau \cos \theta\} \sin^{2n} \theta \, d\theta.$$

If we insert this expression in the right-hand side of (14.6), we have

$$\psi = \int_0^\pi \sum_{n=0}^\infty B_n\left(\frac{\tau}{\mu} \exp\{-i\beta\} \exp\{i\tau \cos \theta\} \sin^2 \theta\right)^n d\theta,$$

where B_n's and μ are arbitrary constants.[6]

Thus, summing up the integrand into the form of an arbitrary function $G(e^\zeta)$, as done in the previous case, we get the general solution of our fundamental equation (14.2) in the form:

$$\psi = \int_0^\pi G(e^\zeta) \, d\theta,$$

with (16.1)

$$\zeta(\theta) = \log \frac{\tau}{\mu} - i\beta + i\tau \cos \theta + 2 \log \sin \theta,$$

where the path of integration from $\theta = 0$ to $\theta = \pi$ should be taken suitably in conformity with remarks which will be given presently in the next section.

We now consider a limiting case in which

$$\left.\begin{array}{l} \tau \rightarrow 0, \\ \mu \rightarrow 0, \end{array}\right\} \frac{\tau}{\mu} \rightarrow \xi(\text{finite}). \tag{16.2}$$

Such a limiting case corresponds evidently to the case of the incompressible fluid flow, and therefore, if we denote the corresponding limiting value of ψ by ψ_{inc}, we have

$$\psi_{\text{inc.}} = 2 \int_0^{\pi/2} G(W \sin^2 \theta) \, d\theta,$$

with (16.3)

$$W = \xi e^{-i\beta}.$$

[5] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press, 1922, p. 48.

[6] Like the previous constant λ, this constant μ is a parameter determining the Mach number of the flow.

We assume that the stream function $\psi_{\text{inc.}}$ for a known incompressible fluid flow is given by

$$\psi_{\text{inc.}} = g(W). \tag{16.4}$$

Then, substituting this in the left-hand side of (16.3), we have an integral equation for determining the arbitrary function G in the form:

$$g(W) = 2\int_0^{\pi/2} G(W \sin^2 \theta)\, d\theta. \tag{16.5}$$

If we put

$$W \sin^2 \theta = \chi,$$

this equation reduces to the form:

$$g(W) = \int_0^W \frac{1}{\sqrt{W - \chi}} \frac{G(\chi)}{\sqrt{\chi}}\, d\chi. \tag{16.6}$$

This is nothing but an integral equation of Abel's type and the solution is obtained by making use of the well-known formula as:

$$\begin{aligned} G(\chi) &= \frac{1}{\pi} \sqrt{\chi} \frac{\partial}{\partial \chi} \int_0^\chi \frac{g(W)}{\sqrt{\chi - W}}\, dW \\ &= \frac{1}{\pi}\left\{ g(0) + \sqrt{\chi} \int_0^\chi \frac{g'(W)}{\sqrt{\chi - W}}\, dW \right\}. \end{aligned} \tag{16.7}$$

Thus, if we put this expression for G into the integrand in the formula (16.1), we shall obtain the stream function for the corresponding flow of a compressible fluid. In the following lines, we shall give a few examples.

(a) As a first example, we consider the case in which

$$\psi_{\text{inc.}} = g(W) = (1 - W)^{-1/2}.$$

Putting this in the above formula (16.7), we readily have

$$G(\chi) = \frac{1}{\pi} \frac{1}{1 - \chi}\,.$$

Thus, inserting this expression for G in (16.1), we have ultimately

$$\psi = \frac{1}{\pi} \int_0^\pi \frac{d\theta}{1 - e^\zeta},$$

with (16.8)

$$\zeta(\theta) = \log \frac{\tau}{\mu} - i\beta + i\tau \cos \theta + 2 \log \sin \theta.$$

(b) As a second example, we next consider the case in which

$$\psi_{\text{inc.}} = g(W) = \log (1 - W).$$

Substituting this expression for $g(W)$ into (16.7), we get

$$G(\chi) = -\frac{2}{\pi}\left(\frac{1}{\chi} - 1\right)^{-1/2} \tan^{-1}\left(\frac{1}{\chi} - 1\right)^{-1/2}.$$

Therefore, by (16.1), we obtain a solution of the form:

$$\psi = -\frac{2}{\pi}\int_0^\pi (e^{-\zeta} - 1)^{-1/2} \tan^{-1} (e^{-\zeta} - 1)^{-1/2}\, d\theta,$$

with (16.9)

$$\zeta(\theta) = \log\frac{\tau}{\mu} - i\beta + i\tau\cos\theta + 2\log\sin\theta.$$

17. Remarks on the path of integration for the integral (16.1). From the above examples it will be seen that when any solution for the incompressible fluid flow reveals singularity at the point $W = 1$, the integrand of the corresponding solution of the integral form (16.1) for the compressible fluid flow has in general a singular point at $\theta = \theta_0$, where θ_0 is a root of the equation:

$$\zeta(\theta) = \log\frac{\tau}{\mu} - i\beta + i\tau\cos\theta + 2\log\sin\theta = 0, \tag{17.1}$$

and the position of such a singular point $\theta = \theta_0$ in the θ-plane varies with the values of τ and β.

Now, it is well known that in case a singular point of the integrand of any function defined by a definite integral moves across the path of integration with the variation of a variable, the said function loses in general its analytic continuity with regard to the variable. Therefore, in order that the solution (16.1) be capable of maintaining its analytic continuity, it is necessary that the path of integration should be such a curve connecting the two points $\theta = 0$ and $\theta = \pi$ which can be deformed, with the variation of τ and β, without being cut across by any singular point of the integrand (Fig. 23).

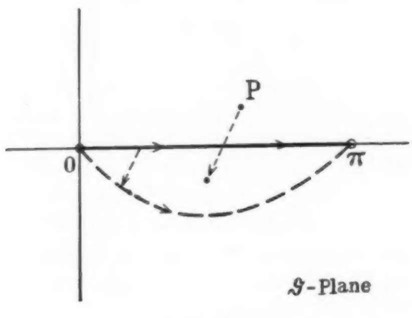

Fig. 23.

18. The singular point of the solution (16.1). Next, we shall discuss the singular point of the solution ψ given by (16.1), which corresponds to the singular point $W = 1$ of the stream function $\psi_{inc.}$ for the incompressible fluid flow.

Now, it is found that Eq. (17.1) has in general two roots, and the corresponding two singular points of the integrand of ψ are usually situated in such a way that the one, denoted by P, lies on one side of the path of integration, while the other, denoted by Q, on the opposite side (Fig. 24). However, for a particular set of values of τ, β, which will be denoted here by τ_∞, β_∞ as before, the confluence of these two singular points

may occur. In this particular case, then, the path of integration must pass through the confluent singular points, and the solution ψ has a singularity.

Thus, it is found that the singular point $(\tau_\infty, \beta_\infty)$ of the solution (16.1) is determined by the two equations $\zeta = 0$ and $d\zeta/d\theta = 0$, namely:

$$\begin{gathered} \log \frac{\tau}{\mu} - i\beta + i\tau \cos \theta + 2 \log \sin \theta = 0, \\ - i\tau \sin \theta + 2 \cot \theta = 0. \end{gathered} \tag{18.1}$$

Eliminating θ from these equations, we obtain the equation for determining τ_∞, β_∞ in the form:

$$\beta_\infty - i \mp \sqrt{\tau_\infty^2 - 1} + i \log \left\{ \frac{2}{\mu \tau_\infty} (1 \mp i\sqrt{\tau_\infty^2 - 1}) \right\} = 0, \tag{18.2}$$

and this equation corresponds to Eq. (15.6) in the preceding analysis.

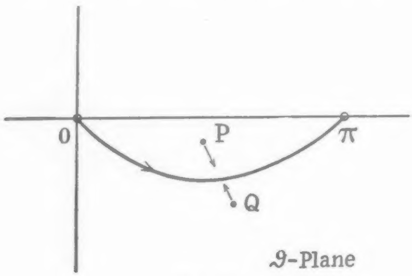

FIG. 24.

After some calculations it can be shown however that this equation becomes in accord with the preceding equation (15.6), if we take the value of the parameter μ equal to $2e^{-1}\lambda$. Therefore, further development of the analysis can be made along the same lines as in the case of Eq. (15.6), and we thus arrive at the conclusion that the singularity of the present solution (16.1) has also the same characteristic features as those of the preceding solution $\psi_{(-1/2)}$ which have been described in detail in §15.

19. The solution giving the uniform flow past an obstacle. In the following lines, we shall discuss the flow past an obstacle by making use of the two fundamental solutions $\psi_{(-1/2)}$ and $\psi_{(1/2)}$ obtained in §15, namely:

$$\psi_{(-1/2)} = \frac{1}{1 - \tau \cos \omega}, \tag{19.1}$$

$$\psi_{(1/2)} = \tau \sin \omega, \tag{19.2}$$

with

$$\beta - i \log \lambda + \tau \sin \omega - \omega = 0. \tag{19.3}$$

We put

$$\omega = r + is.$$

Then, inserting this in the left-hand side of (19.3) and separating the real and imaginary parts, we obtain the relations between τ, β and r, s in the forms:

$$\tau = \frac{s + \log \lambda}{\cos r \sinh s}, \qquad \beta = r - (s + \log \lambda) \tan r \coth s. \tag{19.4}$$

In the first place, we shall examine a solution as expressed by the imaginary part of $\psi_{(-1/2)}$ given by (19.1), namely:

$$\psi = I\{\psi_{(-1/2)}\} = \frac{-(s + \log \lambda) \tan r}{\{1 - (s + \log \lambda) \coth s\}^2 + (s + \log \lambda)^2 \tan^2 r}. \tag{19.5}$$

Putting the denominator equal to zero, we have the equations for determining the coordinates τ_∞, β_∞ of the singular point of ψ in the forms:

$$1 - (s_\infty + \log \lambda) \coth s_\infty = 0,$$

$$(s_\infty + \log \lambda) \tan r_\infty = 0,$$

or

$$\tanh s_\infty = s_\infty + \log \lambda,$$

$$r_\infty = 0.$$

Thus, taking (19.4) into account, we obtain the result that

$$\tau_\infty = \operatorname{sech} s_\infty, \qquad \beta_\infty = 0,$$

and therefore it is seen that the solution ψ under consideration has a singularity at the point $\tau = \tau_\infty = \operatorname{sech} s_\infty$, $\beta = \beta_\infty = 0$.

Also, it will be shown without difficulty that ψ is always equal to zero along the axis $\beta = 0$ from $\tau = 0$ to $\tau = \tau_\infty$.

Along any streamline $\psi =$ const., we can in general express r as a function of s, with the aid of (19.5). Therefore, if we substitute the function $r(s)$ so obtained into Eqs. (19.4), we can obtain the relation between τ and β with s as the parameter. In other words, we can thus obtain the flow pattern in the τ, β plane, which is shown in Fig. 25.

In the next place, we shall consider a solution which is obtained by taking the real part of $\psi_{(1/2)}$ as given by (19.2), namely:

$$\psi = R\{\psi_{(1/2)}\} = \tau \sin r \cosh s. \tag{19.6}$$

It can easily be shown that this solution possesses a branch-point of order 1/2 at the same point ($\tau = \tau_\infty = \operatorname{sech} s_\infty$, $\beta = \beta_\infty = 0$) as the preceding solution (19.5), and that ψ becomes always equal to zero along the axis $\beta = 0$ from $\tau = 0$ to $\tau = \tau_\infty$. In this case, the flow pattern in the τ, β plane becomes as shown in Fig. 26.

From these figures it is naturally expected that if we superpose the above two solutions (19.5) and (19.6) appropriately, we can obtain a solution which would represent the flow pattern in the hodograph plane (i.e., the τ, β plane) as shown in Fig. 27(a), and such a solution would give a required field of flow past an obstacle in the physical plane as shown in Fig. 27(b), as would be expected from the physical meaning of the hodograph plane.

The superposed solution is expressed in the following form:

$$\psi = I\left\{iK\tau \sin \omega + \frac{1}{1 - \tau \cos \omega}\right\},$$

with (19.7)

$$\beta - i \log \lambda + \tau \sin \omega - \omega = 0,$$

where K is a certain constant to be determined appropriately.

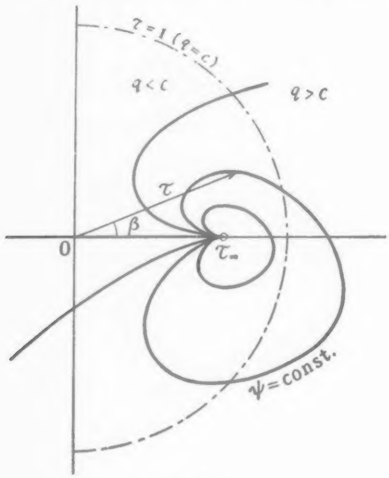

FIG. 25. $\psi_{(-1/2)}$

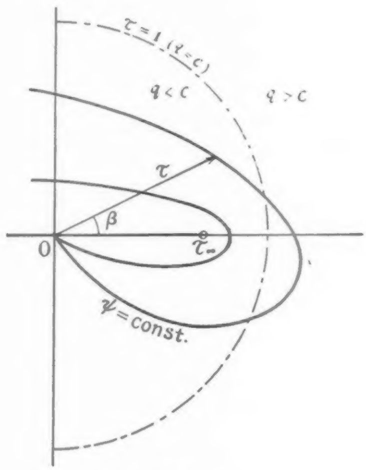

FIG. 26. $\psi_{(1/2)}$

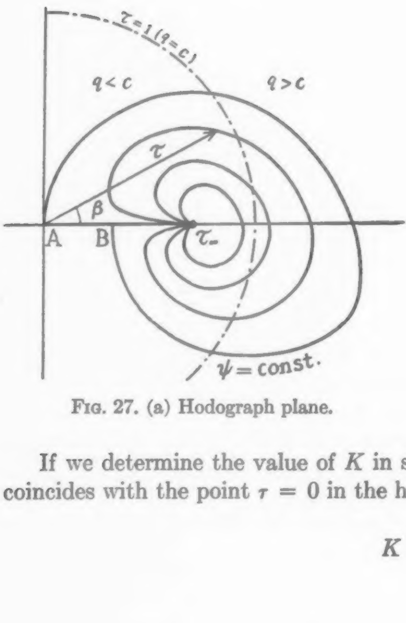

FIG. 27. (a) Hodograph plane.

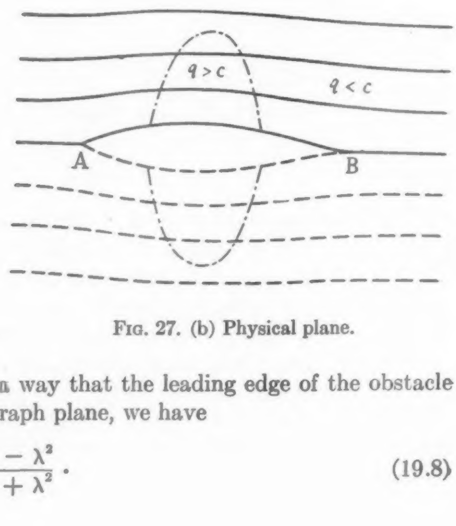

FIG. 27. (b) Physical plane.

If we determine the value of K in such a way that the leading edge of the obstacle coincides with the point $\tau = 0$ in the hodograph plane, we have

$$K = \frac{1 - \lambda^2}{1 + \lambda^2} \cdot \tag{19.8}$$

In this case, it is easily found that the leading edge of the body becomes a stagnation point and that the upper and lower surfaces of the body meet at a finite angle $(2\sqrt{a}/\kappa)\pi$ there.

If, in the fundamental equations (13.1), we change the independent variables from w, θ to τ, β by the aid of (13.5) and (14.1), we have

$$\begin{aligned} \varphi_\tau &= \sqrt{a}\,(\tau - 1/\tau)\psi_\beta\,, \\ \varphi_\beta &= \sqrt{a}\;\tau\psi_\tau\,. \end{aligned} \tag{19.9}$$

Thus, putting the expression (19.7) for ψ into the right-hand sides of these equations and carrying out simple integrations, we obtain the corresponding velocity potential φ in the form:

$$\varphi = \sqrt{a}\tau I\left\{-iK\left(\frac{\tau}{2} + \cos\omega\right) + \frac{\sin\omega}{1 - \tau\cos\omega}\right\}. \tag{19.10}$$

We shall next consider the transformation equations from the τ, β plane to the physical x, y plane. In general, we have

$$dx = x_\varphi\, d\varphi + x_\psi\, d\psi,$$

$$dy = y_\varphi\, d\varphi + y_\psi\, d\psi.$$

Inserting in the right-hand sides of the well-known expressions for x_φ, y_φ, x_ψ, y_ψ as given in Part I, namely:

$$x_\varphi = \frac{1}{q}\cos\theta, \qquad x_\psi = -\frac{1}{\rho q}\sin\theta,$$

$$y_\varphi = \frac{1}{q}\sin\theta, \qquad y_\psi = \frac{1}{\rho q}\cos\theta,$$

and carrying out integrations, we obtain the coordinates x, y on any streamline $\psi =$ const. $\equiv \psi_1$ in the physical plane in the following forms:

$$\begin{aligned} x &= \int_{\varphi_0}^{\varphi}\left(\frac{1}{q}\cos\theta\right)_{\psi=\psi_1} d\varphi - \int_0^{\psi_1}\left(\frac{1}{\rho q}\sin\theta\right)_{\varphi=\varphi_0} d\psi, \\ y &= \int_{\varphi_0}^{\varphi}\left(\frac{1}{q}\sin\theta\right)_{\psi=\psi_1} d\varphi + \int_0^{\psi_1}\left(\frac{1}{\rho q}\cos\theta\right)_{\varphi=\varphi_0} d\psi, \end{aligned} \tag{19.11}$$

where the point $\varphi = \varphi_0$, $\psi = 0$ has been adjusted so as to correspond to the origin of the x, y plane.

In particular, the coordinates on a particular streamline $\psi = 0$, a part of which coincides with the surface of the body, are calculated by the following formulas:

$$\begin{aligned} x &= \int_{\varphi_0}^{\varphi}\left(\frac{1}{q}\cos\theta\right)_{\psi=0} d\varphi, \\ y &= \int_{\varphi_0}^{\varphi}\left(\frac{1}{q}\sin\theta\right)_{\psi=0} d\varphi. \end{aligned} \tag{19.12}$$

Since Eqs. (19.7) and (19.10) give the relations between φ, ψ and τ, β (and consequently, between φ, ψ and q, θ), all the integrands in the formulas (19.11) and (19.12) can be expressed as functions of φ, ψ. Therefore, carrying out integrations, we can obtain the coordinates (x, y) on each of various streamlines and the flow pattern in the physical plane can thus be found.

20. Numerical computations. By assuming $\gamma = 1.4$ for air, detailed numerical computations have been carried out for three cases in which the Mach number M of the

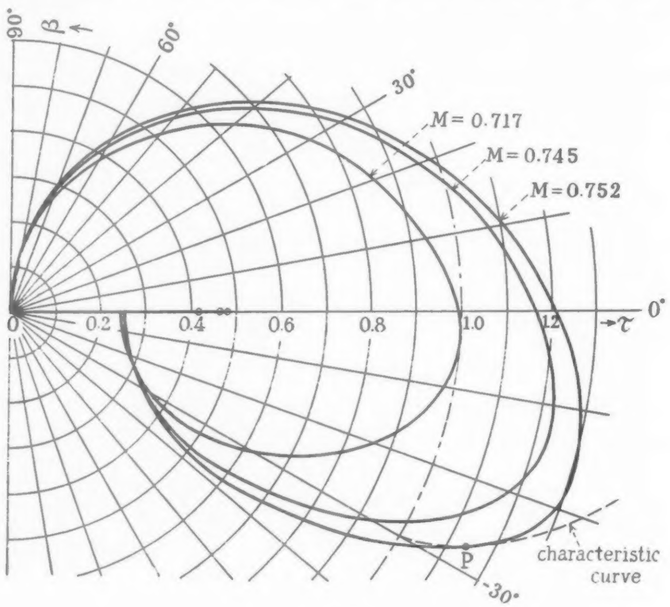

FIG. 28.

undisturbed uniform flow is equal to 0.717, 0.745 and 0.752 respectively, paying special attention to the state of affairs on the surface of the body, the corresponding values of the parameter λ being 0.542, 0.600 and 0.616 respectively. Here, $M = 0.717$ is the so-called critical Mach number at which the maximum local Mach number in the field

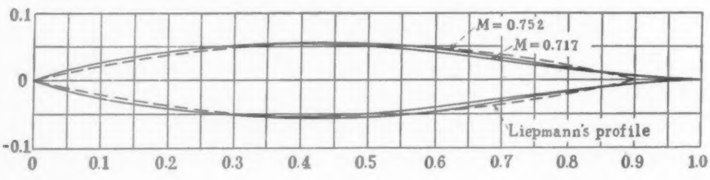

FIG. 29.

of flow becomes equal to unity. Fig. 28 shows the streamlines $\psi = 0$ in the τ, β plane for these three cases.

The coordinates (x, y) of the point on the surface of the obstacle in the physical plane

have been calculated by the formulas (19.12). Denoting the chord length of the body by l, the values of x/l and y/l are shown in Table II. The profiles of the obstacles for the two cases in which $M = 0.717$ and $M = 0.752$ respectively are shown in Fig. 29. The profile for the case $M = 0.745$ has been omitted, however.

It will be seen that there is a very satisfactory coincidence, especially in the forward part as well as in the vicinity of the trailing edge, between the profiles for the three cases, and therefore it may be assumed that the shape of the obstacle is fixed in spite of the variation of the Mach number M of the undisturbed flow.

Values of the fluid velocity q on the surface of the obstacle have been calculated

TABLE II

$M = 0.717$ ($\lambda = 0.542$)			$M = 0.745$ ($\lambda = 0.600$)			$M = 0.752$ ($\lambda = 0.616$)		
x/l	y/l	q	x/l	y/l	q	x/l	y/l	q
0	0	0	0	0	0	0	0	0
0.0299	0.0104	0.686	0.0323	0.0106	0.706	0.0338	0.0111	0.709
0.0874	0.0236	0.793	0.0943	0.0246	0.821	0.0669	0.0191	0.780
0.1399	0.0330	0.854	0.1506	0.0346	0.881	0.1274	0.0310	0.862
0.1897	0.0402	0.898	0.2040	0.0420	0.930	0.1837	0.0396	0.917
0.2376	0.0455	0.927	0.2553	0.0475	0.967	0.2375	0.0460	0.961
0.2843	0.0493	0.951	0.3051	0.0514	0.997	0.2894	0.0506	0.994
0.3299	0.0518	0.972	0.3535	0.0539	1.023	0.3397	0.0537	1.022
0.3747	0.0531	0.989	0.4008	0.0552	1.043	0.3888	0.0556	1.048
0.4000	0.0533	0.997	0.4264	0.0553	1.053	0.4367	0.0561	1.069
0.4190	0.0532	0.999	0.4472	0.0552	1.062	0.4753	0.0555	1.083
0.4632	0.0520	0.988	0.4930	0.0541	1.077	0.5217	0.0538	1.097
0.5086	0.0493	0.947	0.5385	0.0517	1.075	0.5681	0.0506	1.067
0.5563	0.0450	0.896	0.5621	0.0496	0.986	0.5730	0.0501	1.000
0.6065	0.0395	0.853	0.6142	0.0432	0.893	0.5940	0.0475	0.938
0.6590	0.0333	0.817	0.6704	0.0357	0.841	0.6502	0.0397	0.864
0.7136	0.0267	0.785	0.7298	0.0275	0.800	0.7103	0.0311	0.815
0.7704	0.0199	0.757	0.7919	0.0192	0.765	0.7736	0.0223	0.777
0.8293	0.0132	0.729	0.8569	0.0113	0.733	0.8401	0.0137	0.742
0.8904	0.0070	0.704	0.9246	0.0044	0.703	0.9098	0.0060	0.709
0.9536	0.0020	0.682	0.9954	0.0001	0.678	0.9827	0.0005	0.681
0.9861	0.0004	0.672	1.0000	0	0.675	1.0000	0	0.676
1.0000	0	0.667						

and they are given in Table II, and the velocity distributions on the surface of the body are shown in Fig. 30.

It will be seen clearly from Fig. 28 that when $M = 0.752$, the streamline $\psi = 0$ becomes in contact with one of the characteristic curves of the fundamental equation (14.2) at some point P in the τ, β plane. In other words, the singularity $J = \partial(x, y)/\partial(q, \theta) = 0$ makes its first appearance at this Mach number at some point in the field of flow in the hodograph plane, and the velocity gradient becomes infinite at the corresponding singular point P on the surface of the body as shown in Fig. 30. We shall denote by M_s the Mach number at which infinite velocity gradient occurs on the

surface of the body and a trace of the so-called shock line (i.e., the singular point $J = 0$) first appears, and call it provisionally "the shock Mach number" for the sake of convenience. Thus, for our obstacle we have $M_s = 0.752$.

From Fig. 30 it will readily be found that the curve of velocity distribution does not reveal any peculiarity even at the critical Mach number $M = 0.717$. But, it becomes rapidly steeper in the vicinity of the point P as the Mach number increases until, at the above-mentioned shock Mach number, it has an infinite gradient at the point P.

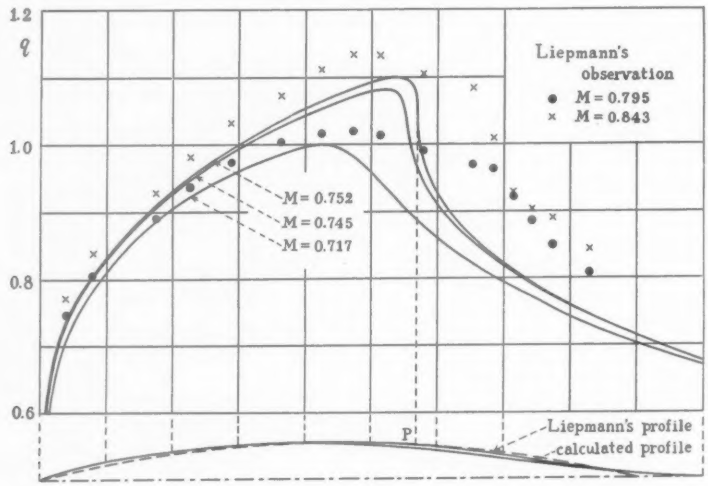

FIG. 30.

For still higher Mach number, two curves of singularity $J = 0$ grow up from the surface of the body. Then, the theoretical field of flow becomes many-valued in the neighbourhood of such curves. At this stage, shock waves must form in the actual field of flow so as to avoid the appearance of the many-valued region which is expected theoretically.

21. A comparison with observation. Quite recently, Hans W. Liepmann[7] has measured the pressure distributions on the surface of a biconvex circular arc profile placed in a high-speed wind tunnel. The dotted-line curve in Fig. 29 shows the biconvex profile of thickness ratio of 0.12 used by Liepmann in his observations, by adjusting it, for the sake of comparison, so as to have the same position of the leading edge as well as the same maximum thickness with those of our profiles derived theoretically, which have been shown by full-line curves.

From Liepmann's observed pressure distributions, we have calculated, by the use of Bernoulli's theorem together with the isentropic law, the velocity distributions on the surface of his obstacle in two cases in which $M = 0.795$ and $M = 0.843$ respectively. The observed values of the velocity q thus found are shown in Fig. 30 by small black circles and crosses respectively. It may be remarked here that at the former Mach number $M = 0.795$, the first appearance of shock waves was observed. Taking account

[7]Hans W. Liepmann, *The interaction between boundary layer and shock waves in transonic flow*, J. Aero. Sci., **13**, 623–637 (1946).

of the discrepancy between the calculated profile and the biconvex profile used in Liepmann's observations, it may be said that the agreement between the theory and observation is satisfactory.

It seems worth noticing here that (a) the observed position of the main shock wave as appeared at an earlier stage falls within the many-valued region derived theoretically in the above and that (b) the main shock wave inclined obliquely forwards as observed by Liepmann, which is enveloping Mach waves starting from the inside of the field of flow but not from the surface of the body, should be just compared with the curve of singularity $J = 0$ found theoretically, which is as well an envelope of one family of Mach waves starting from the inside of the field of flow.

Now, as will be seen from Fig. 30, the theoretical field of flow for the case in which $M = 0.745$, for example, is evidently partially supersonic in limited regions, but is still capable of being continuous and irrotational throughout the whole field of flow. Hence, we arrive at the affirmative positive answer to the so-called Taylor's problem enquiring about whether there is any theoretical possibility of the existence of a continuous irrotational flow of a compressible fluid past an obstacle such that it flows uniformly at a great distance from the body and at the same time contains limited supersonic regions in the neighbourhood of the obstacle; namely, the theoretical results of our analysis show that when a body is placed in a uniform stream of a compressible fluid moving at speeds less than that of sound, the local speed of flow can exceed that of sound in some limited regions in the vicinity of the body without violating the irrotationality as well as the analytical continuity of the flow.

In this connection, it may be emphasized that the solution used here of our fundamental equation (14.2) is expressed in a closed form but not in a form of infinite series and therefore it is quite free from the question of convergence.

Lastly, it may be added that the theoretical results obtained in the above for the flow of our hypothetical gas seem to be still valid, not only qualitatively but also quantitatively, for the flow of the real gas subject to the exact isentropic law, because, in the flow treated above, the maximum speed of flow exceeds the local speed of sound by only about 10 per cent even at the so-called shock Mach number and consequently, our second hypothetical gas as employed in the present Part III can approximate very satisfactorily the real gas obeying the isentropic law throughout the whole field of flow, as is seen clearly from Fig. 21.

MINIMAL PROBLEMS IN AIRPLANE PERFORMANCE*

BY

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Abstract. We develop here the theory of operating an airplane so as to minimize an arbitrary function of the end-values of the generalized coordinates. A propeller-driven airplane is treated as a particle in equilibrium, subject to the forces of drag, lift, thrust, and gravity. We assume that the specific fuel consumption is a function of the power only, and that the available power is independent of the altitude.

The problem is shown to be of the Bolza type in the Calculus of Variations, with the complications arising from the presence of inequalities, discontinuities, and variables whose derivatives do not enter the problem explicitly. The Euler-Lagrange equations are derived and discussed.

Notation. A subscript will sometimes denote an index, at other times the argument of partial differentiation. A superscript dot will indicate differentiation with respect to the parameter t. The Summation Convention will be observed. In referring to equations decimals may be used; e.g. (59.4) is the fourth equation of the set (59). δ_{ij} is the Kronecker delta.

1. Introduction. In the absence of lateral wind we shall treat the airplane as a point, P, in a four-space, specified by the coordinates (T, X, Y, m). Here T is the time, X the length of arc of a great circle of the earth, Y the altitude; and m the mass of the airplane. The end-conditions prescribe the point of departure, P_1, as $T_1 = 0$, $X_1 = 0$, $Y_1 = 0$, $m_1 = m\,(0)$, and some, but not all, of the coordinates of the destination, P_2. We seek to minimize some prescribed function, G, of the remaining coordinates of P_2. The following types of problem are of obvious practical significance:

1) $G = T_2$, minimizing the time of flight,
2) $G = -X_2$, maximizing the range,
3) $G = -Y_2$, maximizing the altitude,
4) $G = -m_2$, minimizing the fuel consumption,
5) $G = -X_2(m_2 - m_{\min})$, maximizing the "transport",
6) $G = -(a + m_2)/T_2$, maxmizing the "profit".

Regardless of the nature of G, and the end-conditions, all problems lead to the same set of the Euler-Lagrange equations. We shall derive the latter from the physical laws governing the motion of an airplane.

2. The physics of the problem. The dynamical laws governing the steady motion of an airplane are

$$\begin{aligned} \tau &= D + mg \sin\phi, \\ L &= mg \cos\phi \end{aligned} \tag{1}$$

where τ, D, L are the thrust, the drag, and the lift, respectively, m the mass, g the accelera-

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tion of gravity, and ϕ the inclination of the trajectory to the horizon. The drag coefficient, C_D and the lift coefficient, C_L, are defined by

$$\left.\begin{aligned} D &= \tfrac{1}{2}C_D S\rho v^2, \\ L &= \tfrac{1}{2}C_L S\rho v^2 \end{aligned}\right\}, \tag{2}$$

S being the "characteristic" area, ρ the density of the air, and v the air speed. C_D and C_L are connected by a parabolic relation[1]

$$C_D = A + C_L^2/B, \tag{3}$$

A and B being a pair of constants.

The power delivered by the propeller is

$$N\epsilon\Pi = \tau v, \tag{4}$$

where N is the number of engines, ϵ the "propulsive" efficiency, and Π the power developed per engine. ϵ is assumed to be an empirical function of the relative air density, σ,

$$\epsilon = \epsilon(\sigma). \tag{5}$$

The specific fuel consumption, C, defined by

$$\frac{dm}{dT} = -NC\Pi, \tag{6}$$

will be assumed to be an empirical function of Π only;

$$C = C(\Pi). \tag{7}$$

We may impose the requirement that the lean fuel mixture be used if $\Pi < \Pi^*$, and the rich mixture if $\Pi > \Pi^*$. At the transition point, Π^*, the function $C(\Pi)$, generally, has a discontinuity.

The distribution of air density will be assumed to obey the exponential law

$$\sigma = \rho/\rho_0 = e^{-Y/\beta}, \tag{8}$$

β being a constant. The effect of wind and the effect of cowl-flaps have been treated in a separate paper, and will not be considered here. A zero wind implies

$$\frac{dX}{dT} = v \cos\phi. \tag{9}$$

The variables Y and Π are bounded by the inequalities:

$$Y \geq 0, \qquad \Pi_{\min} \leq \Pi \leq \Pi_{\max}. \tag{10}$$

The following simplifying assumptions are made:

1) The use of a supercharger makes the range of available power independent of the mode of operation.

2) The trajectory has a gentle slope: $\cos\phi \simeq 1$.

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[1] R. von Mises, *Theory of Flight*, first edition, 142.

3. Choice of variables. We define dimensionless quantities ω, θ, π, c, E:

$$-\omega = \log (m/m_0), \qquad \theta = \log [C_L/(AB)^{1/2}],$$
$$\pi = \Pi/\Pi_0, \qquad c = C/C_0, \qquad E = \epsilon/\epsilon_0. \tag{11}$$

Here m_0 is the initial mass, θ is related to the angle of attack, ϵ_0 is the value of ϵ at $Y = 0$, and (Π_0, C_0) is a pair of values satisfying (7). If C has a single minimum, as is generally the case, we take

$$C_0 = C_{\min}. \tag{12}$$

It is convenient to use the logarithmic variables ξ, η, and the derived variables η', g_0 defined by

$$\xi = \log \pi, \qquad \eta = \log c\pi,$$
$$\eta' = d\eta/d\xi, \qquad (1 - 2g_0)\eta' = 1. \tag{13}$$

We note that at $\pi = 1$, or $\xi = 0$

$$c = 1, \qquad dc/d\pi = 0, \qquad d^2c/d\pi^2 > 0,$$

$$\eta' = 1, \qquad g_0 = 0$$

and

$$\eta' > 0, \qquad g_0 < 1/2 \qquad \text{for all } \xi. \tag{14}$$

The latter follows from the experimental fact that $|\, dm/dT \,|$ in (6) is an increasing function of π.

Dimensionless t, x, y are defined by introducing the scale factors β_0, β_1, β_2:

$$T = \beta_0 t, \qquad X = \beta_1 x, \qquad Y = \beta_2 y. \tag{15}$$

It is convenient to choose

$$\beta_0 = \frac{m_0 g}{N\epsilon_0\Pi_0}\beta,$$
$$\beta_1 = \frac{1}{2}\left(\frac{B}{A}\right)^{1/2}\beta, \tag{16}$$
$$\beta_2 = \beta.$$

Two dimensionless parameters represent the aerodynamic characteristics and the engine characteristics of the airplane:

$$a = gC_0\beta/\epsilon_0$$
$$b = (2m_0g)^{3/2}(S\rho_0)^{-1/2}A^{1/4}B^{-3/4}/N\epsilon_0\Pi_0. \tag{17}$$

Now the system of 11 equations (1-9) in 13 variables X, Y, m, v, Π, C, ϵ, σ, τ, D, L, C_D, C_L reduces to a system of three equations,

$$E\pi e^{\omega} = \dot{x} \cosh \theta + \dot{y},$$

$$\dot{x} = be^{(-\omega+y-\theta)/2}, \tag{18}$$

$$\dot{\omega} = ac\pi e^{\omega},$$

where $E(y) \sim 1$, and can, generally, be represented as

$$E = e^{\kappa y}, \qquad \kappa = \text{const.} \ll 1. \tag{19}$$

In the logarithmic form (18) is equivalent, in view of (13), (19), to

$$\phi_1 = \tfrac{1}{2}(-\omega + y - \theta) - \log \dot{x} + \text{const.} = 0,$$

$$\phi_2 = -\omega - \eta + \log \dot{\omega} + \text{const.} = 0, \tag{20}$$

$$\phi_3 = \log (\dot{x} \cosh \theta + \dot{y}) + \eta - \xi - \kappa y - \log \dot{\omega} = 0,$$

$\eta(\xi)$ being a known function. Thus the five generalized coordinates, y_i , where

$$y_1 = x, \qquad y_2 = y, \qquad y_3 = \omega, \qquad y_4 = \theta, \qquad y_5 = \xi, \tag{21}$$

specify the state of the airplane in terms of its position, mass, the angle of attack, and the power as functions of the time, $y_0 = t$. Three non-holonomic equations of constraint, (20), leave us with two degrees of freedom. These can be realized physically by an arbitrary choice of θ, controlled by the "elevator", and an arbitrary choice of ξ, controlled by the throttle. Among the transfinite set of the pairs of functions $(\theta(t), \xi(t))$ we seek that pair which minimizes a prescribed function $G(t_2 , x_2 , y_2 , \omega_2)$, subject to prescribed end-conditions, and satisfying the differential equations (20).

4. Variational approach. We identify our problem with the Problem of Bolza[2] in the Calculus of Variations in the special form: "Required the arc $y_i(t)$ satisfying the equations

$$\phi_\beta(t; y_i , \dot{y}_i) = 0; \qquad i = 1, \cdots , n; \qquad \beta, j = 1, \cdots , m < n, \tag{22}$$

and the end-conditions

$$\Phi_\alpha(t_2 ; y_i(t_2)) = 0; \qquad \alpha = 1, \cdots , r \leq m,$$
$$t_1 = \text{const.}, \qquad y_i(t_1) = \text{const.}, \tag{23}$$

and minimizing the function $G(t_2 , y_i(t_2))$".

In later sections we shall consider the effect of inequalities of the form $\psi(t, y_i) \geq 0$, and discontinuities of ϕ_β with respect to y_i .

The solution of the problem is obtained by introducing a set of variable Lagrangian multipliers, $\lambda_\beta(t)$, a set of constant multipliers, μ_α , and constructing the auxiliary functions F, Γ, J as

$$F = \lambda_\beta \phi_\beta , \qquad \Gamma = G + \mu_\alpha \Phi_\alpha ,$$
$$J = \Gamma + \int_{t_1}^{t_2} F \, dt. \tag{24}$$

[2] G. A. Bliss, *Lectures in the Calculus of Variations*, 189, e.f.

By differentiation we obtain

$$dJ = [(\Gamma_t + F - \dot{y}_i F_{\dot{y}_i})\, dt + (\Gamma_{y_i} + F_{\dot{y}_i})\, dy_i]_{t_2} + \int_{t_1}^{t_2} \left(F_{y_i} - \frac{d}{dt} F_{\dot{y}_i}\right) \delta y_i \, dt, \tag{25}$$

since P_1 is fixed, and $\delta y = dy - \dot{y}\, dt$. Our problem is equivalent to that of minimizing J; i.e. satisfying

$$dJ = 0, \qquad d^2 J > 0. \tag{26}$$

Thus (25) splits into n Euler-Lagrange equations,[3]

$$F_{y_i} = \frac{d}{dt} F_{\dot{y}} \tag{27}$$

and $m + 1$ Transversality Conditions at t_2,

$$\Gamma_{y_j} + p_j = 0, \qquad j = 0, 1, \cdots, m, \tag{28}$$

where we had set

$$\begin{aligned} p_j &= F_{\dot{y}_j}, \qquad j = 1, \cdots, m, \\ p_0 &= F - \dot{y}_i F_{\dot{y}_i}, \qquad y_0 = t. \end{aligned} \tag{29}$$

We shall further assume that the end-conditions at P_2 separate the variables y_j; i.e.

$$\Phi_\alpha = y_\gamma - \text{const.} = 0, \qquad \alpha = 1, \cdots, r \leq m \tag{30}$$

γ assuming r values in the range $0 \leq r \leq m$. Then μ_α exist, and can be eliminated from (28), yielding $m + 1 - r$ equations

$$G_{y_j} + p_j = 0, \qquad j \neq \gamma, \tag{31}$$

involving only the "free" variables; i.e., such y_j as do not enter the end-conditions at t_2.

The $n + m$ unknowns (y_i, λ_β) are determined by the system of differential equations

$$\begin{aligned} F_{y_i} &= \frac{d}{dt} F_{\dot{y}_i}, \qquad i = 1, \cdots, n = 5, \\ \phi_\beta &= 0, \qquad \beta = 1, \cdots, m = 3, \end{aligned} \tag{32}$$

whose order is generally $2n + m$. In our problem, however, the order is depressed by three circumstances: 1) $\dot{y}_i$ in ϕ_β are soluble for y_i, 2) there are $n - m$ equations $F_{y_k} = 0$, $k = m + 1, \cdots, n$, of (32.1), for the y_i whose derivatives are absent in F, 3) λ_m can be eliminated, since F is a linear homogeneous function of λ_β. The resulting order is $2m - 1$, so that $2m - 1$ initial constants must be furnished. m such constants are $y_j(0) = 0$; $m - 1$ additional constants may be chosen as $g_\beta(0)$, $\beta = 1, \cdots, m - 1$, where we define

$$g_\beta = \lambda_\beta / 2\lambda_m . \tag{33}$$

These parameters. $g_\beta(0)$, of our family of extremals can be determined from the conditions of the particular problem. For, there exist r end-conditions, (30), and $m + 1 - r$ Transversality Conditions, (31), connecting the $m + 1$ quantities t_2, $\lambda_m(t_2)$, $g_\beta(0)$.

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[3] *ibidem*, 202, e.f.

5. Euler-Lagrange equations. An extremal, generally, is compounded of arcs lying in the interior of the admissible region $\psi \geq 0$, and of arcs lying in the boundary $\psi = 0$. In view of the inequalities (10), prescribed in our problem, we must distinguish arcs of the following types:

A) general case, $\pi \not\equiv$ const., $y \not\equiv 0$;
B) flight under constant power, $\pi \equiv$ const., $y \not\equiv 0$;
C) level flight, $\pi \not\equiv$ const., $y \equiv 0$.

These arcs are joined together in accord with the Corner-Condition, discussed in section 8.

In order to take into account inequalities $\psi \geq 0$ we augment F so as to include the constraints $\psi = 0$, by writing

$$\phi_4 = \begin{cases} \xi - \xi_{\min}, \\ \xi_{\max} - \xi \end{cases} \qquad \phi_5 = \tfrac{1}{2}y, \tag{34}$$
$$\lambda_4\phi_4 = 0, \qquad \lambda_5\phi_5 = 0,$$

and construct $\overline{F} = \lambda_\beta\phi_\beta$, $\beta = 1, \cdots, 5$, with the aid of (20), (33):

$$\begin{aligned} \overline{F} = \{(g_1 + g_5 - \kappa)y - (g_1 + 2g_2)\omega - g_1(\theta + 2 \log \dot{x}) + \\ \cdot \log(\dot{x} \cosh \theta + \dot{y}) + (1 - 2g_2)(\eta - \log \dot{\omega}) - (1 \mp 2g_4)\xi\} \end{aligned} \tag{35}$$

The Euler-Lagrange equation in θ is

$$g_1(\dot{x} \cosh \theta + \dot{y}) = \dot{x} \sinh \theta, \tag{36}$$

which, in vew of (18), is equivalent to

$$p = be^{(-3\omega + y)/2}/E, \qquad g_1 = \frac{p}{\pi} e^{-\theta/2} \sinh \theta \tag{37}$$

$p(\omega, y)$ being defined by (37.1). The Euler-Lagrange equations in x, y, ω, ξ can be simplified by eliminating from them $\dot{x}, \dot{y}, \dot{\omega}$ with the aid of (18), and then making use of (37) and (13). On introducing the constant

$$\alpha = \tfrac{1}{2} \log 3 = \coth^{-1} 2, \tag{38}$$

the final result can be written as

$$\begin{aligned} \lambda_3 \sinh(\theta - \alpha)/E\pi e^{\omega} &= \text{const.}, \\ \frac{d}{d\omega} \log(\lambda_3/E\pi e^{\omega}) &= \frac{E}{ac}(g_1 + g_5 - \kappa), \\ \frac{d}{dt}[\lambda_3(1 - 2g_2)/ac\pi e^{\omega}] &= \lambda_3(g_1 + 2g_2), \\ \eta'(g_0 - g_2) \pm g_4 &= 0. \end{aligned} \tag{39}$$

Moreover, since F does not contain t explicitly, there exists the integral $\bar{F} - \dot{y}_i\bar{F}_{\dot{y}_i} =$ const.; i.e.,

$$\lambda_3(g_1 - g_2) = \text{const.}, \tag{40}$$

which we shall use in place of (39.3).

The Lagrange multipliers can be eliminated from (39) as follows. First λ_3 is eliminated among (40), (39.1, 2); next g_4, g_5 are eliminated with the aid of (34). Then (39) becomes

$$\begin{aligned} g_2 &= g_1 + K \sinh(\theta - \alpha)/E\pi e^{\omega}, \\ (g_2 - g_0)(\pi &- \text{const.}) = 0, \\ y\left\{\frac{d}{d\omega}\log\sinh(\theta - \alpha) \right. &+ \left. \frac{E}{ac}(g_1 - \kappa)\right\} = 0, \end{aligned} \tag{41}$$

K being a constant. Finally, g_1 and g_2 are eliminated from (41.1,2), with the result

$$(g_0\pi - pe^{-\theta/2}\sinh\theta - K\sinh(\theta - \alpha)/Ee^{\omega})(\pi - \text{const.}) = 0. \tag{42}$$

Equations (18) can be written, in view of (37), as

$$\begin{aligned} \frac{dt}{d\omega} &= \frac{1}{ac\pi e^{\omega}}, \\ \frac{dx}{d\omega} &= \frac{Epe^{-\theta/2}}{ac\pi}, \\ \frac{dy}{d\omega} &= \frac{E}{ac}\left(1 - \frac{pe^{-\theta/2}}{\pi}\cosh\theta\right). \end{aligned} \tag{43}$$

6. Computational procedure. The airplane is represented by three constants a, b, κ and the function $c(\pi)$. These automatically define the functions

$$c = c(\pi), \qquad g_0 = g_0(\pi), \qquad E = E(y), \tag{44}$$

in view of (13), (19). We recall that g_1, p are known functions of y_i:

$$p = be^{-(3\omega+y)/2}/E(y), \qquad g_1 = \frac{p}{\pi}e^{-\theta/2}\sinh\theta. \tag{37}$$

Thus (41.3), (42), (43) is a definitive system of five equations, and determines t, x, y, θ, π as functions of ω, provided five initial constants are furnished. Three such constants are $t(0) = 0$, $x(0) = 0$, $y(0) = 0$, prescribed by the end-conditions; two additional constants may be chosen as $\theta(0)$ and K, leading to a two-parameter family of extremals

$$y_i = y_i(t; \theta(0), K); \qquad i = 1, \cdots, 5$$

Since t, x do not enter the system explicitly, the corresponding equations (43.1, 2) can be split off, and done by quadrature after the solution for y, θ, π has been obtained.

Thus in the general case, A, $\pi \not\equiv$ const., $y \not\equiv 0$, our system of equations consists of (37), (44), and

$$\frac{dy}{d\omega} = \frac{E}{ac}\left(1 - \frac{pe^{-\theta/2}}{\pi}\cosh\theta\right),$$

$$\frac{d}{d\omega}\log\sinh(\theta - \alpha) + \frac{E}{ac}(g_1 - \kappa) = 0, \tag{45}$$

$$g_0\pi = pe^{-\theta/2}\sinh\theta + K\sinh(\theta - \alpha)/Ee^{\omega}.$$

A choice of $\theta(0)$ and K determines in succession $\pi(0)$ and $g_1(0)$ from (45.3) and (37), so that the integration may proceed.

In case B (45.3) is replaced by $\pi \equiv$ const., and K can be discarded. In case C, $y \equiv 0$, we have $g_1 = \tanh\theta$, and

$$p = be^{-3\omega/2},$$

$$\pi = pe^{-\theta/2}\cosh\theta, \tag{46}$$

$$g_0\pi = pe^{-\theta/2}\sinh\theta + Ke^{-\omega}\sinh(\theta - \alpha).$$

Here a choice of K determines θ and π when y and ω are known. In both the special cases, therefore, the number of initial parameters is reduced to one.

The parameters $\theta(0)$, K for a particular problem can be determined from the end-conditions and the Transversality Condition at t_2. In order to make use of the latter it is necessary to consider first the Sufficiency Condition and the Corner Condition.

7. Sufficiency condition. In the notation of Bliss the sufficiency conditions for a weak relative minimum are I, III′, IV′. Condition I is met by the solution of the Euler-Lagrange equations. The strengthened Legendre-Clebsch Condition,[4] III′, is equivalent to

$$F_{z_\lambda z_\mu}\,\delta z_\lambda\,\delta z_\mu > 0, \tag{47}$$

where $z = (\dot{y}_j, y_k)$; $j = 1, \cdots, m$, $k = m + 1, \cdots, n$; i.e. the set of the highest derivatives entering F, and δz satisfy the differentiated equations of constraint,

$$\phi_{\beta z_i}\,\delta z_i = 0, \qquad \beta = 1, \cdots, m;\ i = 1, \cdots, n. \tag{48}$$

Applying (47) and (48) to (35) and (20), respectively, and eliminating $\delta\dot{y}_i$, we are led to the requirement that

$$\lambda_3\left\{\frac{pe^{\theta/2}}{4\pi}(1 + 3e^{-2\theta})\,\delta\theta^2 + 2\eta'\frac{d}{d\pi}(g_0\pi)\,\delta\xi^2\right\} > 0 \tag{49}$$

be a positive-definite quadratic form. Since p, π, η' are positive, we deduce

$$\left.\begin{aligned}\lambda_3 &> 0,\\ \frac{d}{d\pi}(g_0\pi) &> 0\end{aligned}\right\}. \tag{50}$$

(50.1) is equivalent to

$$\lambda_3(t_2) > 0 \tag{51}$$

[4] *ibidem*, 235.

For, y_i and $\dot{y}_i$ are bounded by physical considerations, and g_1, g_2 are bounded in view of (37) and (41.1). Hence, (39.3) implies that λ_3 cannot change its sign.

(50.2) restricts the range of π. Generally, $g_0\pi$ has a single minimum at some value $\bar{\pi} < 1$. Then we construct a new lower bound, $\pi'_{\min}$ as

$$\pi'_{\min} = \max\,(\pi_{\min}\,,\,\bar{\pi}). \tag{52}$$

As a numerical example, let us take

$$\begin{aligned} c &= 4/(\pi + 1)(3 - \pi), \quad &&\text{if} \quad \pi < \pi^* = 1.5 \quad &&\text{(lean mixture)},\\ c &= 0.5(\pi + 1), &&\text{if} \quad \pi > 1.5 &&\text{(rich mixture)},\\ &&&\pi_{\min} = 0.5, \quad \pi_{\max} = 2.0. \end{aligned} \tag{53}$$

Then

$$\begin{aligned} g_0 &= \pi(\pi - 1)/(\pi^2 + 3), \quad &\frac{dg_0}{d\pi} &= (\pi^2 + 6\pi - 3)/(\pi^2 + 3)^2 \quad &&\text{if} \quad \pi < 1.5,\\ g_0 &= \pi/2(1 + 2\pi), &\frac{dg_0}{d\pi} &= 1/2(1 + 2\pi)^2 &&\text{if} \quad \pi > 1.5, \end{aligned} \tag{54}$$

and (50.2) becomes

$$\pi^2 + 9\pi - 6 > 0$$

so that $\bar{\pi} = 0.638$, and $\pi'_{\min} = 0.638$.

Jacobi's Condition,[5] IV′ has received in our treatment only an empirical verification, in the fact that the family of extremals covers simply the admissible region of space.

8. The Corner condition. Consider an extremal compounded of the branches E_{10} and E_{02} joined at $t = t_0$. If the junction point lies in the surface $\psi(y_i) = 0$, then its coordinates satisfy

$$\psi_{y_i}\, dy_i = 0. \tag{55}$$

Now applying (25) to the two branches, respectively, and adding the results we obtain the additional term

$$\Delta J_0 = -[(\Delta p_j)\, dy_j]_{t_0}\,, \qquad j = 0, 1, \cdots, m. \tag{56}$$

where we define the "jump" in any function, f, as

$$\Delta f = \lim_{\epsilon \to 0}\, [f(t_0 + \epsilon) - f(t_0 - \epsilon)] = f_+ - f_- \tag{57}$$

If J is minimized it is necessary that $\Delta J_0 \geq 0$, subject to (55). This leads to $\Delta J_0 = 0$, and

$$\begin{aligned} \Delta p_j &= \mu\psi_{y_j}\,,\\ 0 &= \mu\psi_{y_k}\,, \end{aligned} \tag{58}$$

μ being a constant, and $p_k \equiv 0$. (58) can be regarded as an extension of the Weierstrass-

[5] *ibidem*, 258.

Erdman Corner Condition.[6] If we now impose the requirement $\Delta y_i = 0$, dictated by physical considerations, we shall have the system of equations

$$\begin{aligned}
\Delta p_j &= \mu \psi_{\dot{y}_j}, \qquad j = 0, \cdots, m, \\
0 &= \mu \psi_{\dot{y}_k}, \\
\Delta y_i &= 0, \\
\phi_\beta &= 0, \qquad \beta = 1, \cdots, m, \\
F_{\dot{y}_k} &= 0, \qquad k = m + 1, \cdots, n.
\end{aligned} \tag{59}$$

The last line gives the Euler-Lagrange equations in the variables y_k.

Suppose E_{10} lies in the admissible region $\psi \geq 0$, and E_{02} lies in the boundary $\psi = 0$. If F is of class C^2 at t_0, and satisfies the Legendre-Clebsh Condition, it can be shown that the only solution of (59) is the trivial solution

$$\Delta y_i = 0, \qquad \Delta \dot{y}_j = 0, \qquad \Delta \lambda_\beta = 0, \qquad \mu = 0. \tag{60}$$

This is the, so-called, Tangency Condition, which requires that all variables and first derivatives appearing explicitly in F be continuous at the junction. Moreover, for the branch E_{02} we have

$$\psi_{y_i}\, \delta y_i \geq 0, \tag{61}$$

which, in view of (25) and $dJ \geq 0$, leads to the Convexity Condition,

$$\begin{aligned}
F_{y_i} - \frac{d}{dt} F_{\dot{y}_i} + \lambda \psi_{y_i} &= 0, \\
\lambda(t) &\leq 0.
\end{aligned} \tag{62}$$

$\lambda(t)$ can be readily identified with the Lagrange multiplier of the constraint $\psi = 0$, entering the augmented function $\mathfrak{F} = F + \lambda\psi$. The point at which $\lambda = 0$ may be termed the point of "inflection".

In our problem there are two inequalities $\psi \geq 0$; (34) gives

$$\psi_1 = \phi_4, \qquad \psi_2 = \phi_5.$$

Then (62.2) requires that $\lambda_4 \leq 0$, $\lambda_5 \leq 0$. Since $\lambda_3 > 0$ by (50), we obtain from (33)

$$g_4 \leq 0, \qquad g_5 \leq 0. \tag{63}$$

Now (39.4), in view of (14), yields

$$\begin{aligned}
g_0 &\geq g_2 \qquad \text{if } \pi = \pi_{\min}, \\
g_0 &= g_2 \qquad \text{if } \pi_{\min} < \pi < \pi_{\max}, \\
g_0 &\leq g_2 \qquad \text{if } \pi = \pi_{\max},
\end{aligned} \tag{63}$$

[6] *ibidem*, 12, 203.

while (39.2, 1) leads to

$$\frac{1}{ac}(g_1 - \kappa) + \frac{d}{d\omega} \log \sinh (\theta - \alpha) \geq 0 \qquad \text{if} \qquad y \equiv 0. \tag{65}$$

The latter determines the point of inflection, beyond which minimal level flight cannot proceed.

Application of the Corner Condition to the discontinuity at $\xi = \xi^*$ will be made in the Appendix.

9. Transversality condition. There are two inequalities affecting y_i at t_2: $y \geq 0$, $\omega_{\max} - \omega \geq 0$. Thus the function G in (31) is to be augmented by writing

$$\begin{aligned} &\overline{G} = G + \mu_\alpha \Phi_\alpha , && \alpha = 1, 2 \\ &\Phi_1 = y, && \Phi_2 = \omega_{\max} - \omega \\ &\mu_1 y = 0, && \mu_2(\omega_{\max} - \omega) = 0. \end{aligned} \tag{66}$$

Then (31) becomes

$$\begin{aligned} G_t - 2\lambda_3 K \sinh (\theta - \alpha)/E\pi e^{\omega} &= 0, \\ G_x - \lambda_3 \sinh (\theta - \alpha)/E\pi e^{\omega} \sinh \alpha &= 0, \\ y(G_y + \lambda_3/E\pi e^{\omega}) &= 0, \\ [G_\omega - \lambda_3(1 - 2g_2)/ac\pi e^{\omega}](\omega_{\max} - \omega) &= 0, \end{aligned} \tag{67}$$

upon elimination of μ_1, μ_2 with the aid of (66).

We shall now consider problems of the type $G = \pm y_r$, r assuming any one of the values $0, \cdots m = 3$. Then $G_{y_i} = \pm \delta_{ri}$. Observing that $a, \alpha, \omega, \pi, c, E$ are positive, we deduce from (67) and (51) the following Table:

Transversality Condition at t_2

Free Variable	If $G = y_i$	If $G = -y_i$	If $G_{y_i} = 0$
$y_0 = t$	$K(\theta - \alpha) > 0$	$K(\theta - \alpha) < 0$	$K(\theta - \alpha) = 0$
$y_1 = x$	$\theta > \alpha$	$\theta < \alpha$	$\theta = \alpha$
$y_2 = y$	—	—	$y = 0$
$y_3 = \omega$	$g_2 < \frac{1}{2}$	$g_2 > \frac{1}{2}$	$g_2 = \frac{1}{2}$ or $\omega = \omega_{\max}$.

In the last column the variable y_i is ignored at t_2; i.e., it is an argument neither of G nor of the end-conditions at t_2.

If the entire extremal is of the type B, $\pi \equiv$ const., then dt and $d\omega$ in (25) are no longer

independent, but connected by $d\omega = ac\pi e^{\omega}\, dt$. As a result, the first and last rows of the Table coalesce into

$$\left.\begin{array}{l} y_0 = t \\ y_3 = \omega \end{array}\right\} \qquad g_1 < \tfrac{1}{2}, \qquad\qquad g_1 > \tfrac{1}{2}, \qquad\qquad g_1 = \tfrac{1}{2} \text{ or } \omega = \omega_{\max} . \tag{68}$$

In applying the Table we make use of the following Remarks concerning minimal curves:

1) "θ and g_1 have like signs"
2) "$K(\theta - \alpha) = 0$ implies $g_1 = g_2$, and conversely"
3) "$\theta \leq \alpha$ and $g_1 > \frac{1}{2}$ implies $\dot{y} < 0$"
4) "$g_2 \geq \frac{1}{2}$ implies $\pi = \pi_{\max}$"
5) "g_2 decreases in passing through the value $\frac{1}{2}$ provided $g_1 > -1$"
6) "The quantities λ_3 , $\theta - \alpha$ and $g_1 - g_2$ cannot change their signs"

The proofs depend on (37), (41.1), (43) and (38), (64) and (14), (39.3), (39.1) and (41.1), respectively. In virtue of the Remarks 6 and 2 the first two lines of the Table hold for all $t < t_2$.

We shall next illustrate the use of the Table by considering a few special cases.

10. Special cases. 1) *Problems not involving the time.*

Here $G_t = 0$, so that $g_1 = g_2$, and $K(\theta - \alpha) = 0$ for all t, there being but a single parameter, say $\pi(0)$. In the case A (45.3) becomes

$$g_0\pi = pe^{-\theta/2} \sinh \theta. \tag{69}$$

In the case C, $y \equiv 0$, (46) reduces to

$$\begin{aligned} \pi &= be^{-(3\omega+\theta)/2} \cosh \theta, \\ g_0 &= \tanh \theta. \end{aligned} \tag{70}$$

Here

$$g_0 = g_1 = g_2 , \tag{71}$$

so that the number of parameters is reduced to zero; the family shrinks to a single curve, which can be calculated by quadratures. For, eliminating θ in (70) and (43) gives

$$\begin{aligned} \omega &= \frac{1}{6} \log \pi^4(1 + g_0)^3(1 - g_0) + \log b^{2/3} \equiv \Omega(\pi) + \log b^{2/3}, \\ t &= b^{-2/3}\tau(\pi) + k_1 , \\ x &= \mathrm{X}(\pi) + k_2 . \end{aligned} \tag{72}$$

τ, X are functionals of $c(\pi)$, and can be tabulated as functions of π. k_1 , k_2 are determined from the initial conditions.

Let us next consider the problem of maximizing the range, x, when t, y, ω are ignored. Then $G = -x$, and the Table gives

$$\begin{array}{llll} \theta < \alpha, & K = g_1 - g_2 = 0 & \text{for all } t, & \\ y = 0, & \omega = \omega_{\max} & \text{or} \quad g_2 = \tfrac{1}{2} & \text{at } t_2 . \end{array}$$

The alternative $g_2 = g_1 = \frac{1}{2}$ at t_2 must be ruled out if $y(0) = 0$. For, then $g_2 = g_1 > \frac{1}{2}$ at $t < t_2$ by Remark 5, and hence $\dot{y} < 0$ by Remark 3, so that the inequality $y \geq 0$ is violated. Thus our solution is the curve for which $y = 0$ when $\omega = \omega_{\max}$.

2) *Problems not involving the range.*

Here $G_x = 0$, so that $\theta = \alpha$ for all t. Then (45) is to be replaced by

$$\frac{dy}{d\omega} = \frac{E}{ac}\left[1 - 2(3)^{-3/4}\frac{p}{\pi}\right],$$

$$(\pi - \pi_{\text{extr}})(g_0\pi - 3^{-3/4}p - K'/Ee^{\omega}) = 0, \tag{73}$$

where the constant $K' = K \sinh(\theta - \alpha)$ need not vanish.

Consider the problem of the most economical climb to a given altitude. We set $G = \omega$, $G_t = G_x = 0$, and note that $\theta = \alpha$, $K' = 0$ for all t. Hence (73.2) becomes

$$(g_0\pi - 3^{-3/4}p)(\pi - \pi_{\text{extr}}) = 0. \tag{74}$$

The first factor has only one root, π_1, since π is a single-valued function of $g_0\pi$ by (50.2). The solution of (74), subject to (64) is the point in the interval $(\pi_{\min}, \pi_{\max})$ nearest to π_1. Having determined π, we proceed with the integration of (73.1), obtaining the unique solution of the problem.

Similarly, the problem of the fastest climb is solved by setting $G = t$, $G_x = G_\omega = 0$, and noting that $\theta = \alpha$, $K' > 0$ for all t, and $g_2 = \frac{1}{2}$ at t_2. Then from the Remarks 1, 5, 4 and the equation (68) it follows that for all t we must have $\pi = \pi_{\max}$, $g_1 < \frac{1}{2}$.

11. Summary. The Euler-Lagrange equations, derived in section 5, can be readily integrated by the procedure of section 6, leading to a two-parameter family of curves. Such a calculation was carried out, for a typical airplane, by the Differential Analyzer at the Ballistic Research Laboratories in 1948, using ten integrators. The complications arising from the presence of inequalities and a discontinuity were easily resolved by applying the Corner Condition, discussed in sections 8, 12.

The family of curves so constructed represents the totality of solutions of all possible problems under consideration. The process of selecting from the family the curve that solves a particular problem is carried out by scanning the end-conditions and the Transversality Condition at t_2. The latter is discussed and illustrated in sections 9, 10. The Sufficiency Conditions for a weak relative minimum can be, generally, satisfied, as shown in section 7.

APPENDIX

12. Effect of discontinuities. Applying the results of section 8 we consider the case in which E_{10} lies in the region $\psi \geq 0$ where $\psi = 0$ is a surface of discontinuity of F with respect to the arguments of ψ. Two possibilities arise:

a) ψ is not a function of y_k, so that $\psi_{y_k} = 0$, and (59.2) drops out. Then the system (59) reduces to $n + 2m + 2$ equations among the $n + 2m + 2$ unknowns $(\Delta y_i, \Delta \dot{y}_i, \Delta\lambda_\beta, \mu)$, and may be expected to have a non-trivial solution at a given $t = t_0$.

b) ψ is a function of y_k, so that (59.2) contributes $\mu = 0$, over-determining the system by one equation. Therefore, at t_0 there exists, generally, only the trivial solution (60). Then the extremal must enter the surface $\psi = 0$ by a continuous transition, and remain in $\psi = 0$ until it reaches some point $t = t_0^*$, where a non-trivial solution exists. At that

point there occurs a discontinuous transition, which could be termed "delayed refraction", or "reflection", according as $\psi = 0$ is crossed or not.

In our problem a discontinuity occurs at the surface

$$\psi = \xi^* - \xi = 0$$

Since $\dot{\xi}$ does not appear explicitly in F, we are dealing here with a transition of type b. The system of equations (59), in view of (75) and (35), reduces to

$$\begin{gathered} \Delta\theta = \Delta(g_1\pi) = \Delta(g_2\pi) = \Delta[(1 - 2g_2)/c] = 0, \\ (g_2 - g_0)(\pi - \pi_{\text{extr}}) = 0. \end{gathered} \tag{76}$$

From the last three equations of (76) we deduce

$$\begin{gathered} \left\{(\pi_-)\frac{c_+ - c_-}{\pi_+ - \pi_-} - \left(\pi\frac{dc}{d\pi}\right)_+\right\}(\pi - \pi_{\text{extr}}) = 0, \\ 2(g_2)_+ = (\pi_-)\frac{c_+ - c_-}{(c\pi)_+ - (c\pi)_-}, \\ (g_2\pi)_+ = (g_2\pi)_- . \end{gathered} \tag{77}$$

which, in conjunction with (64), determines π_+, $(g_2)_+$, $(g_2)_-$. These three values are invariant, being dependent only on the function $c(\pi)$.

As an example, let us suppose that in (53) the discontinuity at π^* is approached from the "lean" branch of the curve. Then $\pi_- = \pi^* = 1.5$, $c_- = 1.067$, and the first factor of (77.1), equated to zero, is

$$1.5\,\frac{0.5(\pi + 1) - 1.067}{\pi - 1.5} - 0.5\pi = 0, \qquad \text{if} \qquad \pi > 1.5,$$

whose root, $\pi = 2.24$, falls outside the admissible range of π. The second factor of (77.1) leads to two solutions:

1) $\pi_+ = \pi_{\max} = 2, \quad c_+ = 1.5, \quad (g_2)_+ = 0.232, \quad (g_0)_+ = 0.200,$

$\pi_+ = \pi_{\min} = 0.5, \quad c_+ = 1.067, \quad (g_2)_+ = 0, \quad (g_0)_+ = 0.077,$

the values of c and g_0 being obtained from (53) and (54). Since only the first one of these solutions satisfies (64), we take

$$\pi_+ = 2, \qquad (g_2)_+ = 0.232, \qquad (g_2)_- = 0.309.$$

After entering the surface $\pi = \pi^* = 1.5$ (lean), the extremal remains there until g_2 reaches the value $(g_2)_-$. At this point, which can be observed by means of (40.1) during the computation, a delayed refraction occurs, π jumping to $\pi_+ = 2$.

HEAT TRANSFER BETWEEN SOLIDS AND GASSES UNDER NONLINEAR BOUNDARY CONDITIONS*

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1. Introduction. In the theory of heat transfer between solids and gasses, it is commonly assumed that the rate of heat exchange across a gas-solid interface is proportional to the difference between the temperature of the solid surface and that of the ambient gas. This assumption is known as Newton's Law of Cooling and it gives rise to a boundary condition of the following general form

$$k\left(\frac{\partial U}{\partial n}\right) = -f\Delta U, \tag{1}$$

where k is the thermal conductivity of the solid; $\partial U/\partial n$ is the thermal gradient at the surface evaluated from the interior in the direction of the outward normal; ΔU is the difference in temperature between the surface and the gas, considered positive when the solid is warmer than the gas; and f is the factor of proportionality, frequently referred to as the film transfer factor.** If f is a constant the above boundary condition is linear. At ordinary temperatures, where most of the heat transfer is due to conduction-convection, f varies but slightly with temperature and it is not a bad approximation to regard it as a constant. At higher temperatures however, most of the heat is transferred by radiation and the film transfer factor varies greatly with temperature. Neglecting conduction and convection, we find from the "fourth power law" that f_r , the film transfer factor due to radiation, is given by

$$f_r = A\epsilon \frac{T_s^4 - T_g^4}{T_s - T_g}, \tag{2}$$

where T_s is the absolute temperature of the solid surface, T_g the absolute temperature of the ambient gas, ϵ the emissivity, and A is a constant depending upon the units of measurement. Even when all the heat exchange is by conduction-convection the film transfer factor, f_c , changes somewhat with temperature but the dependence is much more complicated than equation (2) and is expressed in the form of empirically determined relations between certain dimensionless moduli.†

Henceforth we shall not be concerned with any particular form of the relationship between f and the temperature. The important point is that when the film transfer factor is a function of the temperature, the boundary condition (1) becomes nonlinear. It is our purpose in this paper to investigate a nonlinear boundary value problem under

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**L. M. K. Boelter, V. H. Cherry, H. A. Johnson, *Heat transfer*, University of California Press, pp. IIb-1, IIb-2.

†Boelter, Cherry, and Johnson, *op. cit.*, Chap. XII.

the most general physically significant relationship between the film transfer factor and the temperature.

2. Statement of problem. In order to keep geometrical considerations as simple as possible we shall consider only the semi-infinite solid. To simplify further the non-essential aspects of the problem, we shall assume the gas to be maintained constantly at unit temperature. This will enable us to regard the film transfer factor as a function of the surface temperature alone. Requiring that the initial temperature of the solid be zero throughout, the problem of finding the ensuing temperature-time distribution, $U(x, t)$, in the solid can be formulated as follows:

$$U_t(x, t) = U_{xx}(x, t), \qquad x > 0, t > 0, \tag{3}$$

$$U(x, 0) = 0, \tag{3a}$$

$$-U_x(0, t) = \frac{[1 - U(0, t)]f[U(0, t)]}{k} = G[U(0, t)], \tag{3b}$$

$$| U(x, t) | < M > 1 \qquad x > 0, t > 0. \tag{3c}$$

As usual, the following functions will be assumed to be continuous for the values of x and t indicated:

$$U(x, t) \qquad \text{for} \qquad x \geq 0, t \geq 0, \tag{3d}$$

$$U_x(x, t) \qquad \text{for} \qquad x \geq 0, t > 0, \tag{3e}$$

$$U_{xx}(x, t), U_t(x, t) \qquad \text{for} \qquad x > 0, t > 0. \tag{3f}$$

Equation (3) is the well-known heat flow equation, where the units of time and distance have been so chosen as to make the diffusivity 1. (3b) is the special form of boundary condition (1) corresponding to this particular problem. Observe that $-kU_x(0, t)$ is the rate of heat flow per unit area into the solid from the gas. The function $G[U(0, t)]$, which is proportional to this rate of heat exchange, occurs continually throughout the following work and will be referred to as the input function. Condition (3c) serves the double purpose of restricting the behavior of $U(x, t)$ as x tends toward infinity, and of excluding the possibility of an instantaneous heat source at the surface when $t = 0$.

To complete the statement of the problem, some hypotheses must be made concerning the input function, $G[U(0, t)]$. We know from experience that heat transfer takes place in a continuous manner; that a net exchange of heat takes place between two media only when they are at different temperatures; and that the net rate of heat transfer is a monotone increasing function of the difference in temperature between the two media. Referring to the definition of the input function above, we see that in any physically significant problem the following three hypotheses must hold:

A. $G[U]$ is continuous for all U;

B. $G[U]$ is zero when $U = 1$;

C. $G[U]$ is a monotone decreasing function of U.

In the following work we shall repeatedly invoke the above hypotheses, especially in Sec. 5.

For a brief summary of the principal results obtained, the reader is referred to Sec. 6.

3. Reduction of the problem to a nonlinear integral equation. The temperature distribution in the solid, which is the unknown quantity in the problem stated in Sec. 2, is a function of the two variables, x and t. We can easily show, however, by Duhamel's Principle for example, [1] that $U(x, t)$ is completely determined by the surface temperature, $U(0, t)$. This leads us to expect that the problem admits a more concise formulation in terms of the function of a single variable, $U(0, t)$. We shall effect such a re-formulation by means of the Laplace transform. This well-known transformation will not, of course, eliminate the essentially nonlinear character of the problem but it does restate it in terms of a nonlinear integral equation for $U(0, t)$.

In view of the conditions imposed upon $U(x, t)$ in Sec. 2, it is easily verified that the Laplace transformation with respect to t carries (3), (3b), and (3c) into the following linear ordinary boundary value problem:

$$u_{xx}(x, s) - su(x, s) = 0, \tag{3'}$$

$$u_x(0, s) = -\mathfrak{L}\{G[U(0, t)]\} = -g(s), \tag{3b'}$$

$$| u(x, s) | < \frac{M}{s} \quad \text{for} \quad x > 0. \tag{3c'}$$

The general solution of (3′) is

$$u(x, s) = Ae^{-xs^{1/2}} + Be^{xs^{1/2}},$$

where A and B may depend on s but not on x. (3c′) requires that $B = 0$ and from (3b′) we find that $A = g(s)s^{-1/2}$. Hence we get that

$$u(x, s) = \frac{g(s)}{s^{1/2}} e^{-xs^{1/2}} \tag{4}$$

for the solution of the transformed boundary value problem. Inverting (4) by means of the Borel formula [2] we obtain

$$U(x, t) = \int_0^t \frac{G[U(0, \tau)]}{\pi^{1/2}(t - \tau)^{1/2}} \exp \frac{-x^2}{4(t - \tau)} \, d\tau \tag{5}$$

which expresses $U(x, t)$ in terms of all the values of $U(0, \tau)$ between 0 and t. We can now prove the following theorem:

THEOREM 1. *A necessary condition that a function $y(t)$ should be a surface temperature function for the problem stated in Part II is that it satisfy the following singular, nonlinear integral equation of the Volterra type*

$$y(t) = \int_0^t \frac{G[y(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}} \, d\tau \tag{6}$$

and that

$$| y(t) | \leq M \qquad t > 0. \tag{7}$$

If $| y(t) | \leq 1$, then equation (6) *is also a sufficient condition.* The proof of the theorem is facilitated by the following lemma:

LEMMA I. *Any function, $y(t)$, satisfying both* (6) *and* (7) *is continuous for $t \geq 0$ and $y(0) = lim_{t\to 0^+} y(t) = 0$.*

The proof of Lemma I is a straightforward exercise in elementary calculus and will not be included here.

To establish the necessity condition stated in Theorem 1, we assume that $y(t)$ is a surface temperature function for the problem under consideration. This means that there exists a function $U(x, t)$ satisfying the conditions (3) through (3f) and having the property that $y(t) = \lim_{x\to 0} U(x, t) = U(0, t)$. Applying the continuity condition (3d) to equation (5) we get directly that

$$U(0, t) = \int_0^t \frac{G[U(0, \tau)]}{\pi^{1/2}(t - \tau)^{1/2}} \, d\tau,$$

and since $U(0, t) = y(t)$ we see that (6) must hold. (3c) and (3d) give us (7) immediately. This completes the proof of the necessity.

Turning now to the proof of the sufficiency condition, we assume that $y(t)$ satisfies (6) and (7) and show that the function defined as follows

$$U(x, t) = \int_0^t \frac{G[y(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}} \exp \frac{-x^2}{4(t - \tau)} \, d\tau$$

is a solution of the problem stated in Sec. 2. The proof consists in verifying that $U(x, t)$ defined above satisfies the conditions (3) through (3f). Most of these verifications are trivial and will be omitted. We do include however the proof for condition (3b) which is not completely obvious, and that for (3c) which will make clear why the sufficiency is proved only for $| y(t) | \leq 1$.

Considering first (3b) we differentiate (8) to get

$$U_x(x, t) = -\frac{x}{2\pi^{1/2}} \int_0^t \frac{G[y(\tau)]}{(t - \tau)^{3/2}} \exp \frac{-x^2}{4(t - \tau)} \, d\tau.$$

Since both this integral and that on the right in (8) converge uniformly in t on any interval of the form $0 \leq t \leq T$ for any $x > 0$, the above differentiation is valid. Making the substitution $\xi^2 = x^2/4(t - \tau)$ the above becomes

$$U_x(x, t) = -\frac{2}{\pi^{1/2}} \int_{x/2t^{1/2}}^{\infty} G\left[y\left(t - \frac{x^2}{4\xi^2}\right)\right] e^{-\xi^2} \, d\xi.$$

Form now the following difference

$$\Delta = \left| \int_0^{\infty} G[y(t)] e^{-\xi^2} \, d\xi - \int_{x/2t^{1/2}}^{\infty} G\left[y\left(t - \frac{x^2}{4\xi^2}\right)\right] e^{-\xi^2} \, d\xi \right|.$$

We wish to show that for any given fixed $t > 0$ one can find a δ such that $| x | < \delta \rightarrow \Delta < \epsilon$ where ϵ is arbitrarily small. To this end we consider the following inequality

$$\Delta \leq \Delta_1 + \Delta_2 + \Delta_3 ,$$

where

$$\Delta_1 = \left| \int_{x/2t^{1/2}}^{\theta} \left\{ G\left[y\left(t - \frac{x^2}{4\xi^2}\right)\right] - G[y(t)] \right\} e^{-\xi^2} \, d\xi \right|,$$

$$\Delta_2 = \left| \int_{\theta}^{\infty} \left\{ G[y(t)] - G\left[y\left(t - \frac{x^2}{4\xi^2}\right)\right] \right\} e^{-\xi^2} \, d\xi \right|,$$

$$\Delta_3 = \left| \int_0^{x/2t^{1/2}} G[y(t)] e^{-\xi^2} \, d\xi \right|.$$

By the continuity of G and y (see Hypothesis I and Lemma I) we know that there exists a bound $\bar{G}$, such that $\Delta_1 < 2\bar{G}\theta$. Hence, when $\theta = \epsilon/6\bar{G}$

$$\Delta_1 < \epsilon/3, \qquad \text{for } x < \epsilon/3\bar{G}\, t^{1/2}.$$

Similarly, there exists a number η such that

$$\frac{x^2}{4\xi^2} < \eta \rightarrow \left| G[y(t)] - G\left[y\left(t - \frac{x^2}{4\xi^2}\right)\right] \right| < \frac{\epsilon}{3}.$$

On the interval $\theta \leq \iota < \infty$,

$$\frac{x^2}{4\xi^2} \leq \frac{x^2}{4\theta^2}$$

$$\frac{x^2}{4\theta^2} < \eta \Rightarrow x < 2\theta\eta^{1/2} = \frac{\epsilon}{3\bar{G}}\, \eta^{1/2}.$$

By taking $x < (\epsilon/3\bar{G})\eta^{1/2}$ we have

$$\Delta_2 < \frac{\epsilon}{3} \int_0^\infty e^{-\xi^2}\, d\xi < \frac{\epsilon}{3}.$$

Turning now to Δ_3 we can write

$$\Delta_3 < \int_0^\theta G[y(t)]e^{-\xi^2}\, d\xi < \bar{G} \int_0^{\epsilon/3\bar{G}} e^{-\xi^2}\, d\xi < \frac{\epsilon}{3}.$$

Hence, if we choose δ to be the smaller of $\epsilon/3\bar{G}\, \eta^{1/2}$ and $\epsilon/3\bar{G}\, t^{1/2}$ we get

$$x < \delta \Rightarrow \Delta < \epsilon$$

This implies that

$$\lim_{x \to 0} U_x(x, t) = -\lim_{x \to 0} \frac{2}{\pi^{1/2}} \int_{x/2t^{1/2}}^\infty G\left[y\left(t - \frac{x^2}{4\xi^2}\right)\right] e^{-\xi^2}\, d\xi$$

$$= \frac{-2}{\pi^{1/2}} \int_0^\infty G[y(t)]e^{-\xi^2}\, d\xi = \frac{-2}{\pi^{1/2}} G[y(t)] \int_0^\infty e^{-\xi^2}\, d\xi.$$

Therefore,

$$U_x(0, t) = -G[y(t)]$$

Considering now condition (3c) we recall from Hypotheses II and III that $|\, y(t)\, | \leq 1$ implies $G[y] \geq 0$. Making use of the fact that $G[y]$ does not change sign on the interval of integration, we can invoke the theorem of the mean for definite integrals [3] to get

$$|\, U(x, t)\, | = \int_0^t \frac{G[y(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}} \exp \frac{-x^2}{4(t - \tau)}\, d\tau = \exp \frac{-x^2}{4(t - \tau_1)} \int_0^t \frac{G[y(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}}\, d\tau,$$

where $0 < \tau_1 < t$.
By (6) this becomes

$$U(x, t) = \exp\left[\frac{-x^2}{4(t - \tau_1)}\right] y(t) < y(t) \leq 1.$$

The remaining steps in the proof of Theorem 1 are somewhat lengthy but not difficult and we shall leave them to the reader.

The result of our work so far has been to show that any solution of equation (6)

which is bounded in absolute value by 1 will yield, through formula (8), a solution of the nonlinear boundary value problem stated in Part II. We shall see later that when the input function, $G[y]$, satisfies a Lipschitz condition on the unit interval, the nonlinear integral equation (6) and the differential boundary value problem are completely equivalent, i.e. every solution of one will give a solution of the other. From now on we shall be concerned exclusively with equation (6) which we shall refer to as the surface temperature equation.

4. The linear case. Before studying the surface temperature equation in its most general form, we shall consider briefly the important special case in which it is linear. This case arises when the film transfer factor is assumed to have a constant value, say f_0. The input function $G[y(t)]$, then becomes

$$\frac{[1 - y(t)]f_0}{k} = h_0[1 - y(t)]$$

and equation (6) reduces to

$$y(t) = h_0 \int_0^t \frac{1 - y(\tau)}{\pi^{1/2}(t - \tau)^{1/2}}\, d\tau. \tag{6*}$$

The above equation is easily solved by the Laplace transform and gives

$$y(t) = \frac{h_0 t^{1/2}}{\Gamma(3/2)} - h_0^2 \int_0^t e^{h^2_0\tau} \operatorname{erfc}\,(h\tau^{1/2})\, d\tau, \tag{9}$$

$$\text{where } \operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = 1 - \frac{2}{\pi^{1/2}} \int_0^x e^{-\lambda^2}\, d\lambda = \frac{2}{\pi^{1/2}} \int_x^\infty e^{-\lambda^2}\, d\lambda.$$

In the more general case where (6) is nonlinear, the Laplace transform will no longer be useful, since the transform of the product of two functions cannot be expressed in terms of the transforms of the individual functions. For this reason, we solve (6*) by a different method which can be carried over to the nonlinear case. This second method is that of successive approximations. We begin with the function $y_0(t) \equiv 0$ as the first approximation to $y(t)$. We then construct the sequence of functions $\{y_n(t)\}$ defined by the recursive formula

$$y_{n+1}(t) = h_0 \int_0^t \frac{1 - y_n(\tau)}{\pi^{1/2}(t - \tau)^{1/2}}\, d\tau. \tag{10}$$

Making use of the formula

$$\int_0^t \frac{\tau^{n/2}}{[\pi(t - \tau)]^{1/2}}\, d\tau = \frac{\Gamma(n/2 + 1)}{\Gamma(n/2 + 3/2)}\, t^{(n+1)/2}, \tag{11}$$

we obtain

$$y_1(t) = \frac{h_0}{\Gamma(3/2)}\, t^{1/2},$$

$$y_2(t) = \frac{h_0}{\Gamma(3/2)}\, t^{1/2} - \frac{h_0^2}{\Gamma(2)}\, t,$$

$$y_3(t) = \frac{h_0}{\Gamma(3/2)}\, t^{1/2} - \frac{h_0^2}{\Gamma(2)}\, t + \frac{h_0^3}{\Gamma(5/2)}\, t^{3/2},$$

etc.

In this way we are led to the series

$$y(t) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{h_0^k}{\Gamma(k/2 + 1)} t^{k/2} \tag{12}$$

which obviously converges uniformly on any interval of the form $0 \leq t \leq T$. We can therefore verify that $y(t)$ as given by (12) satisfies (6*) by substituting it into that equation and interchanging the order of summation and integration.

We shall see in Part V that the above well-known procedure, with a slight modification, can be successfully applied to the nonlinear surface temperature equation.

5. The nonlinear integral equation. In this section we shall study the following singular, nonlinear, integral equation of the Volterra type

$$y(t) = \int_0^t \frac{G[y(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}} d\tau. \tag{6}$$

We showed in Part III that it must be satisfied by the surface temperature, $U(0, t)$, of the problem stated in Part II, and we therefore refer to it as the surface temperature equation. If we can find a solution of (6) which is bounded in absolute value by 1, then by formula (8) we can construct a solution, $U(x, t)$ for the problem stated in Part II. Furthermore, if (6) can be shown, under certain conditions on G, to have a unique solution and this solution is bounded in absolute value by 1, then it will follow that under these same conditions the original heat transfer problem has a unique solution. All our results will be obtained by exploiting Hypotheses A, B, and C of Part II and it is important that the reader always keep these clearly in mind.

First we shall prove an existence theorem stating that (6) always has at least one solution which is bounded in absolute value by 1. To this end we introduce the following functional transformation

$$\mathfrak{J}[z](t) = \int_0^t \frac{G^*[z(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}} d\tau, \tag{13}$$

where

$$G^*[z(t)] = \begin{cases} G[z(t)] & \text{when} \quad z(t) \leq 1, \\ 0 & \text{when} \quad z(t) > 1. \end{cases}$$

Since $G[1] = 0$, $G^*[z]$ is continuous. Now pick $T > 0$ arbitrarily large and regard it as fixed. Henceforth we shall confine our attention to the interval $0 \leq t \leq T$.

Consider the sequence of functions defined by the recursive formula

$$z_{n+1}(t) = \mathfrak{J}[z_n](t), \qquad \text{where} \qquad z_0(t) \equiv 0. \tag{14}$$

We can evaluate $z_1(t)$ explicitly since $G[z_0(t)]$ has the constant value $G[0]$ which we shall denote by G_0. We get

$$z_1(t) = \frac{G_0}{\Gamma(3/2)} t^{1/2},$$

$z_2(t)$ cannot be evaluated without knowing the input function, $G[y]$, but the following lemma enables us to infer some important facts about the behavior of the sequence $\{z_n(t)\}$.

Lemma II. If $u(t)$ and $v(t)$ are two functions which are continuous for $t \geq 0$, which satisfy the inequality $u(t) > v(t)$ for all $t > 0$, and if $u(0) < 1$, then

$$\mathfrak{J}[u](t) < \mathfrak{J}[v](t), \qquad \text{for all} \qquad t > 0.$$

The above lemma is a very simple consequence of the monotone decreasing nature of $G[y]$ stated in Hypothesis C. Since $z_0(0) = z_1(0) = 0$ and since $z_1(t) > z_0(t)$ for all $t > 0$, it follows from Lemma II by the method of induction that the following inequalities hold for all $t > 0$.

$$z_1(t) > z_3(t) > \cdots > z_{2n+1}(t) > \cdots \tag{15}$$

$$z_0(t) < z_2(t) < \cdots < z_{2n}(t) < \cdots \tag{16}$$

and also

$$z_{2n+1}(t) > z_{2k}(t), \tag{17}$$

where n and k are independent positive integers or zero. Since $z_0(t) \equiv 0$, it follows from the above inequalities that for all $n > 0$, $z_n(t) > 0$ for all $t > 0$. The first few of these functions are sketched in the figure below.

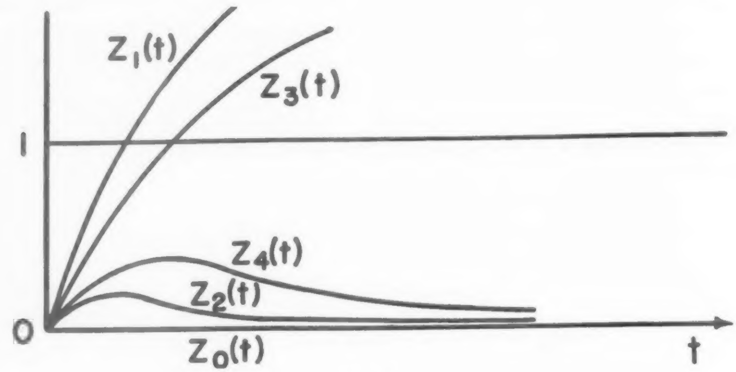

Fig. 1.

All of these functions are bounded from below by $z_0(t)$ and from above by $z_1(t)$. Therefore, on the arbitrary fixed interval $[0, T]$, they are equibounded. We shall find shortly that the $\{z_n(t)\}$ are also equicontinuous. From this it follows by Arzela's theorem that the closure of the set $\{z_n(t)\}$ in the usual topology is compact. It is easy to see, further, that the $\{z_{2n+1}(t)\}$ converge uniformly from above to some function, say $u(t)$, and the $\{z_{2n}(t)\}$ converge uniformly from below to a function, $v(t)$. If $u(t) \equiv v(t)$, then we have a fixpoint of the transformation (13). If $u(t)$ and $v(t)$ are not the same function, then each will be a fixpoint of the square of the transformation (13) but neither will be a fixpoint of the transformation itself.

In order to prove that the transformation (13) does have a fixpoint, we enlarge our attention to the collection of all functions which are continuous on the arbitrarily large fixed interval $[0, T]$. We consider these functions as elements of a Banach space, $\mathfrak{C}$, by

introducing the usual topology, i.e. we denote the norm of an element f by $\|f\|$ and define it as follows

$$\|f\| = \text{l.u.b.} \mid f(t) \mid \qquad \text{on} \qquad [0, T].$$

By the distance between two elements f and g we mean the norm of their difference $\|f - g\|$, and by convergence in $\mathfrak{C}$ we mean uniform convergence. It is easily verified that these definitions do make $\mathfrak{C}$ a Banach space, i.e., a normed linear complete space.

In this space we consider the set Z consisting of all members of $\mathfrak{C}$ which are non-negative, bounded above by $z_1(t)$ and which have a modulus of continuity given by

$$\mid z(t_1) - z(t_2) \mid \leq \frac{4G_0}{\pi^{1/2}} (t_1 - t_2)^{1/2}.$$

It is easy to show that Z is convex and compact.

We shall now show that the transformation carries H into itself. From Lemma II it is easily seen that the transform under (13) of any member of $\mathfrak{C}$ which is contained between $z_0(t)$ and $z_1(t)$ must also lie within these bounds. Moreover, assuming $t_2 > t_1$ and denoting $\mathfrak{Z}[z](t)$ by $u(t)$, we shall have

$$\mid u(t_2) - u(t_1) \mid = \left| \int_0^{t_2} \frac{G^*[z(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}} \, d\tau - \int_0^{t_1} \frac{G^*[z(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}} \, d\tau \right|$$

$$\leq \left| \int_0^{t_1} \frac{G^*[z(\tau)]}{\pi^{1/2}} [(t_2 - \tau)^{-1/2} - (t_1 - \tau)^{-1/2}] \, d\tau \right| + \left| \int_{t_1}^{t_2} \frac{G^*[z(\tau)]}{\pi^{1/2}(t_2 - \tau)^{1/2}} \, d\tau \right|$$

$$< \left| \frac{2G_0}{\pi^{1/2}} [t_2^{1/2} - (t_2 - t_1)^{1/2} - t_1^{1/2}] \right| + \frac{2G_0}{\pi^{1/2}} (t_2 - t_1)^{1/2}$$

since $G^*[z(t)] \leq G[0]$ for all $t > 0$, and $G[0] = G_0$. From here we find that

$$\mid u(t_2) - u(t_1) \mid < \frac{4G_0}{\pi^{1/2}} (t_2 - t_1)^{1/2}.$$

This means that $u(t)$ has the same modulus of continuity as that prescribed for the members of Z. In other words, we have shown that $\mathfrak{Z}$ carries H into a subset of itself, namely Z. Moreover, it is easily seen that $\mathfrak{Z}$ is continuous on H, since if u and v are any two members of H, then

$$\mid \mathfrak{Z}[u](t) - \mathfrak{Z}[v](t) \mid = \left| \int_0^t \frac{G^*[u(\tau)] - G^*[v(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}} \, d\tau \right|$$

$$= \left| \int_0^t \frac{G[u^*(\tau)] - G[v^*(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}} \, d\tau \right|$$

where

$$u^*(t) = \begin{cases} u(t) & \text{when} \quad u(t) \leq 1 \\ 1 & \text{when} \quad u(t) > 1 \end{cases} \tag{18}$$

From this and the continuity of G, it is very simple to show that $\mathfrak{Z}$ is continuous on H.

Summarizing the above results, we have that $\mathfrak{Z}$ is a functional transformation which carries a convex, compact set H of the Banach space $\mathfrak{C}$ into itself continuously. Using

Schauder's generalization from Euclidian space to Banach space of the Brouwer fixpoint theorem, [5] we can infer that $\mathfrak{J}$ has a fixpoint in H. This result will be stated in the following theorem.

THEOREM 2. *On any interval of the form* $[0, T]$ *there is at least one continuous function,* $y(t)$, *such that* $0 < y(t) < (G_0 t^{1/2}/\Gamma(3/2))$ *for all* $t > 0$ *and such that* $y(t) = \mathfrak{J}[y](t)$.

In order to prove the existence of a solution of equation (6), we must prove the following theorem.

THEOREM 3. *If* $y(t)$ *is a continuous fixpoint of the transformation* (13) *for* $0 \leq t \leq T$, *then* $y(t) \leq 1$ *for* $0 \leq t \leq T$.

To prove this, we shall assume that the theorem is false and arrive at a contradiction. Let S be the set of points on the open interval $0 < t < T$ for which $y(t) > 1$. Since $y(t)$ is continuous, S is open. In fact, S consists of disjoint open intervals. Now define

$$y^*(t) = \begin{cases} y(t) & \text{when} \quad t \not\in S, \\ 1 & \text{when} \quad t \in S. \end{cases}$$

The continuity of $y^*(t)$ follows from that of $y(t)$, and obviously

$$\mathfrak{J}[y](t) = \mathfrak{J}[y^*](t). \tag{19}$$

Let t be an arbitrary point of S. Then t belongs to an open interval of the form $t_1 < t < t_2$, and

$$\mathfrak{J}[y](t) = \int_0^{t_1} \frac{G^*[y^*(\tau)]}{[\pi(t-\tau)]^{1/2}}\,d\tau + \int_{t_1}^{t} \frac{G^*[y^*(\tau)]}{[\pi(t-\tau)]^{1/2}}\,d\tau.$$

But since $y^*(t) = 1$ and $G[y^*(t)] = 0$ when $t_1 \leq t \leq t_2$, and since $G^*[y^*] = G[y^*]$ is never negative we can write

$$\mathfrak{J}[y](t) = \int_0^{t_1} \frac{G[y^*(\tau)]}{\pi^{1/2}(t-\tau)^{1/2}}\,d\tau + \int_{t_1}^{t} \frac{G[y^*(\tau)]}{\pi^{1/2}(t-\tau)^{1/2}}\,d\tau = \int_0^{t_1} \frac{G[y^*(\tau)]}{\pi^{1/2}(t-\tau)^{1/2}}\,d\tau$$

$$\therefore\ \mathfrak{J}[y](t) < \int_0^{t_1} \frac{G[y^*]\,d\tau}{\pi^{1/2}(t_1-\tau)^{1/2}} = 1.$$

This means that $\mathfrak{J}[y^*](t) < 1$ on S. But since $\mathfrak{J}[y^*](t) = \mathfrak{J}[y](t)$, we infer that $y(t) = \mathfrak{J}[y](t) < 1$ on S. This is a contradiction unless S is empty and thus the theorem is proved.

We now notice that for any function, $f(t)$, which does not exceed 1, $G[f(t)] = G^*[f(t)]$ and for continuous f

$$\mathfrak{J}[f](t) = \int_0^t \frac{G^*[f(\tau)]}{\pi^{1/2}(t-\tau)^{1/2}}\,d\tau = \int_0^t \frac{G[f(\tau)]}{\pi^{1/2}(t-\tau)^{1/2}}\,d\tau.$$

Since we have just shown that the transformation $\mathfrak{J}$ always has at least one continuous fixpoint, $y(t)$, which is bounded between 0 and 1, $y(t)$ has the following property

$$y(t) = \mathfrak{J}[y](t) = \int_0^t \frac{G^*[y(\tau)]}{\pi^{1/2}(t-\tau)^{1/2}}\,d\tau = \int_0^t \frac{G[y(\tau)]}{\pi^{1/2}(t-\tau)^{1/2}}\,d\tau. \tag{20}$$

We have now obtained the following existence theorem.

THEOREM 4. *For any input function, $G[y]$, there is, on any interval of the form $0 \leq t \leq T$, at least one continuous surface temperature function, $y(t)$, satisfying equation (6) and having the property that $0 < y(t) \leq 1$.*

By virtue of Theorem 1, this means that the nonlinear differential boundary value problem stated in Part II always has a solution.

Theorem 4 expresses the best result which we have been able to obtain with the most general input function, i.e. with the input function subject only to Hypotheses A, B, and C of Part II. After strengthening Hypothesis A, we shall proceed to give a method for obtaining a solution of the surface temperature equation; to prove the uniqueness of this solution; and to find other properties of the surface temperature. This procedure of deriving increasingly better results by progressively strengthening our hypotheses is of mathematical rather than of physical interest, since the strongest hypothesis which we shall ever use in place of A, namely that G is analytic, can safely be assumed in all heat transfer problems of physical significance.

THEOREM 5. *If $G[u]$ satisfies a Lipschitz condition on the closed unit interval $0 \leq u \leq 1$, then the set of approximating functions, $\{z_n(t)\}$ defined by the recursive formula (14) converge to a solution of (6) for all $t > 0$ and the convergence is uniform on every finite interval. If we denote this solution by $y(t)$, then $0 < y(t) \leq 1$ for all $t > 0$.*

The proof is elementary, but since we shall need the same method, with slight variations, in following demonstrations, we shall go through it once here and merely refer to it later.

In view of the fact that $G^*[u(t)] = G[u^*(t)]$ where $u^*(t)$ is defined as in (18), and since $z_n^*(t)$ is between 0 and 1 for all $t \geq 0$ we can write

$$| z_2(t) - z_1(t) | = \int_0^t \frac{| G[z_1^*(\tau)] - G[z_0^*(\tau)] |}{[\pi(t-\tau)]^{1/2}}\, d\tau < L \int_0^t \frac{| z_1^*(\tau) - z_0^*(\tau) |}{[\pi(t-\tau)]^{1/2}}\, d\tau,$$

where L is the Lipschitz constant. Since $| z_1^*(t) - z_0^*(t) | < 1$ for all $t > 0$, this becomes

$$| z_2(t) - z_1(t) | < L \int_0^t \frac{d\tau}{\pi^{1/2}(t-\tau)^{1/2}} = \frac{L}{\Gamma(3/2)}\, t^{1/2}.$$

Similarly,

$$| z_3(t) - z_2(t) | = \int_0^t \frac{| G[z_2^*(\tau)] - G[z_1^*(\tau)] |}{\pi^{1/2}(t-\tau)^{1/2}}\, d\tau < L \int_0^t \frac{| z_2^*(\tau) - z_1^*(\tau) |}{\pi^{1/2}(t-\tau)^{1/2}}\, d\tau.$$

It is easily seen that

$$| z_{k+1}^*(t) - z_k^*(t) | \leq | z_{k+1}(t) - z_k(t) |.$$

Therefore,

$$| z_2^*(t) - z_1^*(t) | < \frac{L}{\Gamma(3/2)}\, t^{1/2}$$

Substituting this into the above bound for $| z_3(t) - z_2(t) |$ gives

$$| z_3(t) - z_2(t) | < \frac{L^2}{\Gamma(3/2)} \int_0^t \frac{\tau^{1/2}}{\pi^{1/2}(t-\tau)^{1/2}}\, d\tau = \frac{L^2}{\Gamma(3/2)} \frac{\Gamma(3/2)}{\Gamma(2)}\, t = \frac{L^2}{\Gamma(2)}\, t.$$

Continuing in this way with the help of formula (11) we get

$$| z_{2n+1}(t) - z_{2n}(t) | < \frac{L^{2n}}{\Gamma(n+1)} t^n.$$

This difference approaches 0 as $n \to \infty$ for any value of t. By the inequalities (15), (16), and (17) it is therefore clear that on any finite interval the functions $\{z_n(t)\}$ converge uniformly toward a limit function, $y(t)$. By virtue of the uniformity of convergence we can pass to the limit under the integral sign as $n \to \infty$ in the recursive formula (14) and in this way show that $y(t)$ must be a fixpoint of the transformation $\mathfrak{I}$. But we proved in Theorem 3 that any continuous fixpoint of $\mathfrak{I}$ is bounded above by 1. This implies that $y(t)$ satisfies (20) and is therefore a solution of (6).

Extending the Lipschitz condition to hold beyond the unit interval we prove the uniqueness of the surface temperature function.

THEOREM 6. *If $G[u]$ satisfies a Lipschitz condition on the interval* $[0, 1 + \epsilon]$ *for any ϵ greater than zero, then equation* (6) *has a unique bounded solution.*

We already know from Theorem 4 that (6) has at least one solution, $y(t)$ such that $0 < y(t) \leq 1$ for all $t > 0$. Let $u(t)$ be any other bounded solution of (6). We know by Lemma 1 that $u(t)$ is continuous and that $u(0) = 0$. Since the integrand of (6) is positive when $u < 1$, we also know that the origin is not a limit point of zeros of $u(t)$. Suppose that $u(t)$ has zeros to the right of the origin and let t_0 be the first one. Let t_k be the first value of t on the interval $0 < t < t_0$ for which $y(t) = 1 + \epsilon/k$ for any positive integer k. Then for $0 < t \leq t_k$ we have $0 < u(t) \leq 1 + \epsilon$ and thus the Lipschitz condition will hold for G. Applying the method of successive approximations illustrated in the proof of Theorem 5 to the difference $u(t_k) - y(t_k)$ we obtain

$$| u(t_k) - y(t_k) | < \frac{(1+\epsilon)L^n}{\Gamma(n/2+1)} t_k^{n/2} \qquad \text{for all } n. \tag{21}$$

Since this difference tends to zero as n approaches infinity, we see that $u(t_k) = y(t_k) \leq 1$ which is a contradiction. Hence there is no value of t on the interval $0 < t < t_0$ for which $u(t)$ exceeds 1. But here again is a contradiction because it is impossible that $u(t_0) = 0$ since by Hypotheses B and C there can have been no negative contribution to the integral in (6). This means that $u(t)$ remains between 0 and 1. Therefore we can use the method of successive approximations to show that (21) holds for all t and hence $u(t) \equiv y(t)$.

THEOREM 7. *If $G[u]$ is analytic on the closed unit interval* $[0, 1]$, *then equation* (6) *has a unique bounded solution, $y(t)$, and $y(t)$ is analytic for all $t > 0$.*

The uniqueness follows from the fact that since G is analytic on the closed unit interval, it can be analytically continued onto a somewhat larger interval $[0, 1 + \epsilon]$. G obviously satisfies a Lipschitz condition on this larger interval and by Theorem 6 we have the uniqueness of the solution.

The proof of the analyticity, which is somewhat lengthy, divides itself into the following three steps. First we shall show how the solution, $y(t)$, of (6) can be represented as the limit of a certain uniformly convergent sequence $\{u_n(t)\}$. Second, we shall show how $y(t)$ can be represented as the limit of a uniformly convergent sequence, $\{w_n(t)\}$, of analytic functions of the real variable, t. Third, we consider t to be a complex variable

and show that a sequence, $\{W_n(t)\}$, of analytic functions of the complex variable converge uniformly to a function, $Y(t)$, which coincides with $y(t)$ along the real axis.

We begin the first step by introducing a set of functions defined as follows:

$$u_0(t) \equiv 0,$$

$$u_{n+1}(t) = \mathfrak{J}_\epsilon[u_n](t) = \int_0^t \frac{G_\epsilon[u_n(\tau)]}{[\pi(t-\tau)]^{1/2}}\,d\tau = \int_0^t \frac{G[\bar{u}_n(\tau)]}{[\pi(t-\tau)]^{1/2}}\,d\tau, \tag{22}$$

$$G_\epsilon[u_n(t)] = \begin{cases} G[u_n(t)], & \text{when} \quad u_n(t) \le 1+\epsilon, \\ G[1+\epsilon], & \text{when} \quad u_n(t) > 1+\epsilon, \end{cases}$$

and where

$$\bar{u}_n(t) = \begin{cases} u_n(t), & \text{when} \quad u_n(t) \le 1+\epsilon, \\ 1, & \text{when} \quad u_n(t) > 1+\epsilon. \end{cases}$$

We see immediately that when $\epsilon = 0$, the sequence $\{u_n(t)\}$ reduces to the sequence $\{z_n(t)\}$ already introduced by the recursive formula (14). Also, $u_1(t) = z_1(t) = G_0/\Gamma(3/2)t^{1/2}$ for any ϵ, but $u_2(t)$, unlike $z_2(t)$, will eventually fall below the t-axis for any $\epsilon > 0$ because of the negative contribution to the integral in (22) when the argument

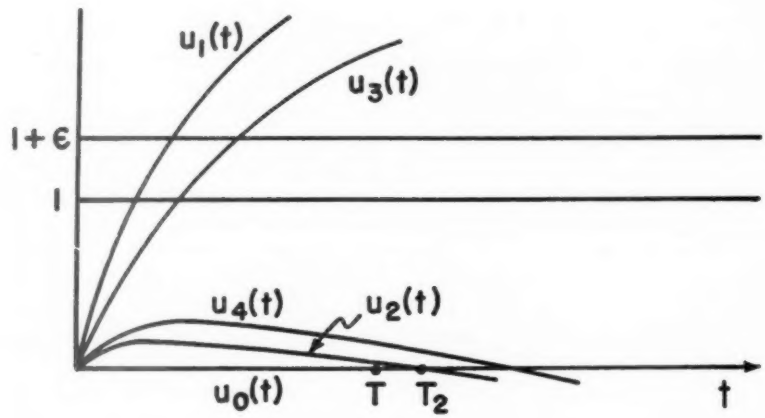

FIG. 2.

of the input function is greater than 1. Let us denote by T_2 this zero of $u(t)$. It is a simple matter to verify the intuitively obvious fact that T_2 is a continuous function of ϵ which decreases from ∞ to some positive number $T_2(0)$ as ϵ increases from 0 to ∞. Confining our attention as usual to a fixed arbitrarily large interval $[0, T]$ on the t-axis we fix ϵ at a constant value sufficiently small that $T_2(\epsilon) > T$ and that $1 + \epsilon$ lies within the interval of analyticity of G.

Recalling the monotone decreasing nature of G, and making use of the fact that $u_2(t) > u_0(t)$ for $0 < t \leq T$, we can easily verify the following inequalities for $0 < t \leq T$:

$$u_1(t) > u_3(t) > u_5(t) > \cdots > u_{2n+1}(t) > \cdots, \tag{23}$$

$$u_0(t) < u_2(t) < u_4(t) < \cdots < u_{2n}(t) < \cdots \tag{24}$$

and

$$u_{2n+1}(t) > u_{2k}(t), \tag{25}$$

where n and k are independent positive integers or zero. Using these inequalities and the method demonstrated in the proof of Theorem 5, we can prove that the sequence $\{u_n(t)\}$ converges uniformly on $[0, T]$ to a fixpoint, $y(t)$, of the transformation $\mathfrak{I}_\epsilon[u](t)$ defined in (22).

We must now show that $y(t)$ is the solution of (6) on the interval under consideration. From the above inequalities we know that $y(t) > 0$ for $0 < t \leq T$. Assume now that t_1 is the first value of t for which $y(t) = 1 + \epsilon$. Then on the closed interval $[0, t_1]$ $G_\epsilon[y(t)] = G[y(t)]$ and $y(t)$ is therefore the solution of (6) on this subinterval. But this means that $y(t) \leq 1$ and contradicts the assumption that $y(t_1) = 1 + \epsilon$. Hence $y(t)$ is the solution of (6) on the entire interval $[0, T]$.

The functions $\{u_n(t)\}$ are not analytic because of the truncating process employed in their definition. We shall now represent their limit function, $y(t)$, as the limit of a uniformly convergent sequence of analytic functions. Choose N so large that

$$n > N \to 0 < u_n(t) < 1 + \epsilon/2 \qquad \text{for} \qquad 0 \leq t \leq T \tag{26}$$

and pick k so large that $2k > N$. It is our purpose to interpose an analytic function, say $w_0(t)$, between $u_{2k}(t)$ and $u_{2k+2}(t)$ on $0 < t \leq T$. By inequalities (23), (24), and (25), and by reasoning similar to that on which Lemma 2 is based, it is easily seen that the successive transforms of $w_0(t)$ under $\mathfrak{I}_\epsilon$ will lie between those of $u_{2k}(t)$ and $u_{2k+2}(t)$. If we designate these successive transforms by $\{w_n(t)\}$ it is therefore clear that $\lim_{n\to\infty} w_n(t) = y(t)$ uniformly on $[0, T]$. Moreover, these functions will be analytic on the open interval $0 < t < T$, since none will be truncated in the recursive formula. From (22) and (26) we see that their recursive definition could be written as follows:

$$w_{n+1}(t) = \int_0^t \frac{G[w_n(\tau)]}{[\pi(t - \tau)]^{1/2}}\, d\tau. \tag{27}$$

In other words, each $w_n(t)$ is the fractional integral of order 1/2 of a function analytic for $t > 0$.

The $u_n(t)$ themselves are analytic on the open interval from the origin up to the point $t = \Gamma[(3/2)(1 + \epsilon)/G_0]^2 = T_\epsilon$ where $u_1(t)$ is truncated. On this interval $[0, T_\epsilon]$ the $\{u_n(t)\}$ have the same recursive definition as that for the $\{w_n(t)\}$ shown in (27) except that $u_0(t) \equiv 0$. In the neighborhood $0 < t < T$, $u_{2k}(t)$ and $u_{2k+2}(t)$ can be expanded in powers of $\sqrt{t}$ as follows.

$$u_{2k}(t) = a_1 t^{1/2} + a_2 t + a_3 t^{3/2} + \cdots + a_n t^{n/2} + \cdots,$$

$$u_{2k+2}(t) = b_1 t^{1/2} + b_2 t + b_3 t^{3/2} + \cdots + b_n t^{n/2} + \cdots.$$

Suppose that $a_i = b_i$ for $i = 1, 2, 3, \cdots, m$ and that $a_{m+1} \neq b_{m+1}$.

Clearly $b_{m+1} > a_{m+1}$. Now form the following functions

$$U_{2k}(t) = \frac{u_{2k}(t) - f_m(t)}{t^{(m+1)/2}}, \qquad U_{2k+2}(t) = \frac{u_{2k+2}(t) - f_m(t)}{t^{(m+1)/2}},$$

where

$$f_m(t) = \sum_{i=1}^{m} a_i t^{i/2} = \sum_{i=1}^{m} b_i t^{i/2}.$$

This gives two functions, $U_{2k+2}(t)$ and $U_{2k}(t)$, such that $U_{2k+2}(t) > U_{2k}(t)$ for $0 \leq t \leq T$. There will be a minimum distance, say δ, between the two functions on $[0, T]$. Approximate to the mean of the two functions by closer than $\delta/2$ with a polynomial, $P(t)$. Then we can write

$$U_{2k}(t) < P(t) < U_{2k+2}(t), \qquad 0 \leq t \leq T,$$

$$t^{(m+1)/2}U_{2k}(t) < t^{(m+1)/2}P(t) < t^{(m+1)/2}U_{2k+2}(t), \qquad 0 < t \leq T,$$

$$t^{(m+1)/2}U_{2k}(t) + f_m(t) < t^{(m+1)/2}P(t) + f_m(t) < t^{(m+1)/2}U_{2k+2}(t) + f_m(t), \qquad 0 < t \leq T.$$

But

$$t^{(m+1)/2}U_{2k}(t) + f_m(t) = u_{2k}(t) \qquad \text{and} \qquad t^{(m+1)/2}U_{2k+2}(t) + f_m(t) = u_{2k+2}(t).$$

Hence if we denote $t^{(m+1)/2}P(t) + f_m(t)$ by $w_0(t)$ we have

$$u_{2k}(t) < w_0(t) < u_{2k+2}(t) \qquad \text{for} \qquad 0 < t \leq T$$

and $w_0(t)$ is analytic on this interval. Using $w_0(t)$ in the recursive formula (27) we get the desired sequence of analytic functions, $\{w_n(t)\}$ converging uniformly to $y(t)$.

Now consider t to be a complex variable, $t = u + iv$. The functions $\{w_n(t)\}$ will

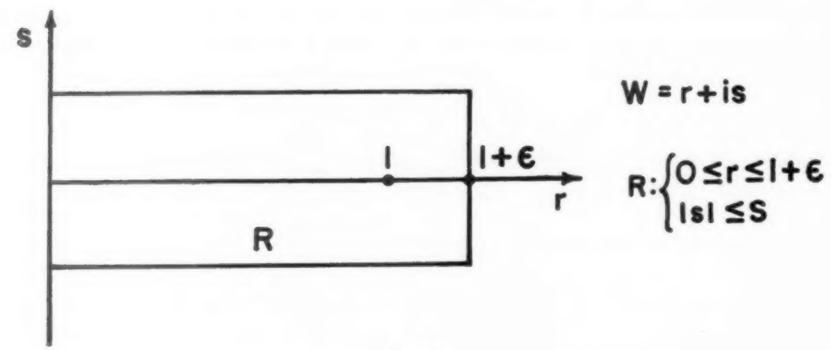

FIG. 3.

then become complex functions which we shall denote by $\{W_n(t)\}$ where $W_n = r_n + is_n$. Since G is analytic on the closed interval $[0, 1 + \epsilon]$ we know that it has an analytic continuation into a neighborhood of this interval in the complex W-plane. In particular we know that there is a closed domain, R, of the form shown below on which G is analytic.

Since G is analytic on R which is closed, we know that there exists a number, D, such that

$$W \varepsilon R \rightarrow | G[W] | < D, \tag{28}$$

and since $W_0(t)$ is a polynomial in powers of $\sqrt{t}$, we know that it is analytic in any region of the complex t-plane which does not include the origin. Consider now a region Δ in the t-plane of the form shown below

$$\Delta : \begin{cases} 0 < u < T \\ | v | < V \end{cases}$$

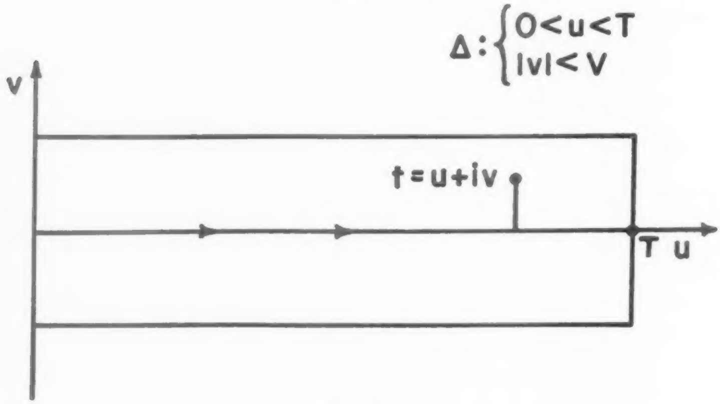

FIG. 4.

$W_0(t)$ can be regarded as a transformation which maps the region, Δ, conformally from the t-plane into the W-plane. Recalling (26) and the construction of $w_0(t)$, we see that $W_0(t)$ carries the segment $[0, T]$ into a subset of $[0, 1 + \epsilon/2]$ on the r-axis. From this and the continuity of $W_0(t)$, it follows that by taking V sufficiently small, say $V < V_1$, $W_0(t)$ will map Δ into R, i.e.

$$t\epsilon\Delta \rightarrow W_0(t)\epsilon R \rightarrow | G[W_0(t)] | < D.$$

It is now our purpose to show that by taking Δ sufficiently narrow,

$$t\epsilon\Delta \rightarrow W_n(t)\epsilon R, \qquad \text{for all } n. \tag{29}$$

Integrating along the path shown in Figure 4 we have

$$W_1(t) = \int_0^u \frac{G[W_0(\tau)]}{[\pi(t - \tau)]^{1/2}}\, d\tau + \int_u^t \frac{G[W_0(\tau)]}{[\pi(t - \tau)]^{1/2}}\, d\tau.$$

The real part of $W_1(t)$ can be estimated as follows

$$\Re\{W_1(t)\} < \int_0^u \frac{G[W_0(\tau)]}{\pi^{1/2}(u - \tau)^{1/2}}\, d\tau + \left| \int_u^t \frac{G[W_0(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}}\, d\tau \right|$$

$$< 1 + \frac{\epsilon}{2} + D \int_u^{u+iv} \frac{d\tau}{\pi^{1/2}(u + iv - u - i\xi)^{1/2}}.$$

Along the vertical strip, $\tau = u + i\xi$ and $d\tau = id\xi$

$$\mathfrak{R}\{W_1(t)\} < 1 + \frac{\epsilon}{2} + D\int_0^v \frac{d\xi}{\pi^{1/2}(v-\xi)^{1/2}} = 1 + \frac{\epsilon}{2} + \frac{D}{\Gamma(3/2)} v^{1/2}.$$

This shows that for all values of t belonging to Δ, we have

$$\mathfrak{R}\{W_1(t)\} < 1 + \frac{\epsilon}{2} + \frac{D}{\Gamma(3/2)} V^{1/2}.$$

By requiring that

$$V < \left[\frac{\epsilon\Gamma(3/2)}{2D}\right]^2 = V_2 ,$$

we insure that $D/\Gamma(3/2)V^{1/2} < \epsilon/2$ and therefore $\mathfrak{R}\{W_1(t)\} < 1 + \epsilon$.

Turning now to the imaginary part, we observe that along the stretch $[0, u]$,

$$\mathfrak{I}\{(t-\tau)^{-1/2}\} < \frac{(2v)^{1/2}}{2[(u-\tau)^2 + v^2]^{1/2}},$$

$$\therefore \mathfrak{I}\{W_1(t)\} < \frac{v^{1/2}}{(2\pi)^{1/2}} \int_0^u \frac{G[W_0(\tau)]}{[(u-\tau)^2 + v^2]^{1/2}} d\tau + \frac{D}{\Gamma(3/2)} v^{1/2}$$

$$< \frac{G_0 v^{1/2}}{(2\pi)^{1/2}} [\ln \{(u^2 + v^2)^{1/2} + u\} - \ln v] + \frac{D}{\Gamma(3/2)} v^{1/2}.$$

Thus, for any t belonging to Δ, the following inequality holds

$$\mathfrak{I}\{W_1(t)\} \leq \frac{G_0 V^{1/2}}{(2\pi)^{1/2}} [\ln \{(T^2 + V^2)^{1/2} + T\} - \ln V] + \frac{D}{\Gamma(3/2)} V^{1/2}.$$

Since the first term on the right-hand side of this inequality tends to zero with V, there exists a number, V_3, such that

$$V < V_3 \rightarrow |\, \mathfrak{I}\{W_1(t)\} \,| < S.$$

Therefore, by taking V to be smaller than the least of the three numbers V_1, V_2, V_3, we define a region Δ, in the complex t-plane which is mapped by the function $W_1(t)$ into the region R in the complex W-plane. By induction it is easy to show that each one of the functions $W_n(t)$ maps Δ into R.

We have now shown that $\{W_n(t)\}$ is a sequence of functions which are analytic in Δ, equibounded in Δ, and which converges uniformly on the interval $0 < t \leq T$, which is an infinite subset of Δ, to the function $y(t)$. By Vitali's theorem we can conclude that $\{W_n(t)\}$ converges uniformly throughout Δ to a function, $Y(t)$, which is the analytic continuation of $y(t)$ into the complex t-plane. $y(t)$ is therefore an analytic function of the real variable t for $0 < t < T$. This completes the proof of Theorem 7, since T was chosen arbitrarily large.

THEOREM 8. *If $G[y]$ is analytic on the unit interval $0 \leq y \leq 1$, then $y(t)$ is monotone increasing for all $t > 0$.*

By differentiating equation (6) we get

$$y'(t) = \int_0^t \frac{G'[y(\tau)]y'(\tau)}{[\pi(t-\tau)]^{1/2}}\,d\tau + \frac{G_0}{(\pi t)^{1/2}}. \tag{30}$$

Perhaps the easiest way of justifying the above result of differentiating the singular integral in the surface temperature equation with respect to the parameter t is to observe that (6) can be written $y(t) = G[y(t)]*1/(\pi t)^{1/2}$ where the star indicates the convolution or faltung operation. Then, by a well-known theorem on the differentiation of the faltung under hypotheses which are satisfied by the above functions [6], formula (30) follows immediately.

$y(t)$ is a power series in $t^{1/2}$ and is analytic for $t > 0$, but whereas by Lemma 1 $y(0) = 0$, $y'(t)$ approaches infinity like $t^{-1/2}$ as t approaches 0. If $y'(t)$ ever becomes negative, there must be a first zero, say t_0, to the right of which is an open interval, I, on which $y'(t)$ is less than 0. Let t_1 be any point belonging to I. Then we must have

$$y'(t_0) = \int_0^{t_0} \frac{G'[y(\tau)]}{\pi^{1/2}(t_0-\tau)^{1/2}}\,y'(\tau)\,d\tau + \frac{G_0}{(\pi t_0)^{1/2}} = 0, \tag{31}$$

$$y'(t_1) = \int_0^{t_0} \frac{G'[y(\tau)]y'(\tau)}{\pi^{1/2}(t_1-\tau)^{1/2}}\,d\tau + \int_{t_0}^{t_1} \frac{G'[y(\tau)]y'(\tau)}{\pi^{1/2}(t_1-\tau)^{1/2}}\,d\tau + \frac{G_0}{(\pi t_1)^{1/2}}.$$

The latter equation can be written as follows

$$y'(t_1) = \int_0^{t_0} \left(\frac{t_0-\tau}{t_1-\tau}\right)^{1/2} \frac{G'[y(\tau)]}{\pi^{1/2}(t_0-\tau)^{1/2}}\,y'(\tau)\,d\tau$$

$$+ \int_{t_0}^{t_1} \frac{G'[y(\tau)]}{\pi^{1/2}(t_1-\tau)^{1/2}}\,y'(\tau)\,d\tau + \frac{G_0}{(\pi t_0)^{1/2}}\,\frac{t_0^{1/2}}{t_1^{1/2}}.$$

In the first integral on the right, the factor $[(t_0-\tau)/(t_1-\tau)]^{1/2}$ decreases monotonically from $t_0^{1/2}/t_1^{1/2}$ to zero. By the theorem of the mean we know that there exists a number τ_1 between 0 and t_0 such that

$$y'(t_1) = \left(\frac{t_0-\tau_1}{t_1-\tau_1}\right)^{1/2} \int_0^{t_0} \frac{G'[y(\tau)]}{\pi^{1/2}(t_0-\tau)^{1/2}}\,y'(\tau)\,d\tau + \int_{t_0}^{t_1} \frac{G'[y(\tau)]y'(\tau)}{\pi^{1/2}(t_1-\tau)^{1/2}}\,d\tau + \frac{t_0^{1/2}}{t_1^{1/2}}\,\frac{G_0}{(\pi t_0)^{1/2}}.$$

Since $G'[y]$ is negative for all y, and since $y'(t)$ has been assumed to be negative on the interval I, the second integral above must be positive. Comparing this last expression for $y'(t)$ with equation (31) and noticing that

$$\left(\frac{t_0-\tau_1}{t_1-\tau_1}\right)^{1/2} < \frac{t_0^{1/2}}{t_1^{1/2}},$$

it is obvious that $y'(t_1) < 0$. This contradiction shows that $y'(t)$ cannot become negative, and since $y'(t)$ is analytic for $t > 0$, we know that $y(t)$ must be monotone increasing.

THEOREM 9. *Let G_L be the space of input functions $G[u]$ which satisfy a Lipschitz condition with constant $<L$ on an interval of the form $0 \le u \le 1 + \epsilon$ for any $\epsilon > 0$. Let Y be the space of corresponding surface temperature functions, $y(t)$. As a metric we take the norm of the difference. The equation* (6) *represents a continuous implicit functional transformation from G_L to Y.*

By Theorem (6) we know that for each element G of G_L there is a unique element $y(t)$ of Y, and that $0 \leq y(t) \leq 1$ for all $t \geq 0$. Select arbitrarily any member G_1 of G_L and let $y_1(t)$ be its corresponding element in Y. Let G_2 and $y_2(t)$ be any other corresponding pair of functions. The difference between the two surface temperature can be written as follows

$$y_2(t) - y_1(t) = \int_0^t \frac{G_2[y_2(\tau)] - G_2[y_1(\tau)]}{\pi^{1/2}(t-\tau)^{1/2}}\,d\tau + \int_0^t \frac{G_2[y_1(\tau)] - G_1[y_1(\tau)]}{\pi^{1/2}(t-\tau)^{1/2}}\,d\tau.$$

If we specify that G_2 be chosen in such a way that $|\,G_2[u] - G_1[u]\,| < \delta$ for $0 \leq u \leq 1$, then a bound for the above difference is shown below:

$$|\,y_2(t) - y_1(t)\,| < L \int_0^t \frac{|\,y_2(\tau) - y_1(\tau)\,|}{\pi^{1/2}(t-\tau)^{1/2}}\,d\tau + \int_0^t \frac{\delta}{\pi^{1/2}(t-\tau)^{1/2}}\,d\tau. \tag{32}$$

In view of the fact that $|\,y_2(t) - y_1(t)\,| < 1$ for all $t > 0$, (32) becomes

$$|\,y_2(t) - y_1(t)\,| < L \int_0^t \frac{d\tau}{\pi^{1/2}(t-\tau)^{1/2}} + \delta \int_0^t \frac{d\tau}{\pi^{1/2}(t-\tau)^{1/2}} = \frac{L+\delta}{\Gamma(3/2)}\,t^{1/2}.$$

Substituting this bound back into (32) we get

$$|\,y_2(t) - y_1(t)\,| < \frac{L(L+\delta)}{\Gamma(2)}\,t + \frac{\delta}{\Gamma(3/2)}\,t^{1/2}.$$

By repeating this process of successive approximations n times we get

$$|\,y_2(t) - y_1(t)\,| < \frac{\delta t^{1/2}}{\Gamma(3/2)} + \frac{L\delta}{\Gamma(2)}\,t + \frac{L^2\delta}{\Gamma(5/2)}\,t^{3/2} + \cdots + \frac{L^n\delta}{\Gamma((n+3)/2)}\,t^{(n+1)/2}$$
$$+ \frac{L^{n+1}(L+\delta)}{\Gamma(n/2+2)}\,t^{n/2+1}.$$

Letting n tend toward infinity, the above can be written as follows

$$|\,y_2(t) - y_1(t)\,| < L\delta t\left[\frac{1}{\Gamma(2)} + \frac{L^2t}{\Gamma(3)} + \frac{L^4t^2}{\Gamma(4)} + \frac{L^6t^3}{\Gamma(5)} + \cdots\right]$$
$$+ \delta t^{1/2}\left[\frac{1}{\Gamma(3/2)} + \frac{L^2t}{\Gamma(5/2)} + \frac{L^4t^2}{\Gamma(7/2)} + \frac{L^6t^3}{\Gamma(7/2)} + \cdots\right].$$

Restricting ourselves to an arbitrarily large fixed interval $[0, T]$ we can now show that

$$|\,y_2(t) - y_1(t)\,| < \delta T^{1/2}[1 + LT^{1/2}]e^{L^2T}.$$

For any given L and T the above difference can be made arbitrarily small by taking δ sufficiently small. This completes the proof of Theorem 9.

Making use of the fact that any monotone continuous function, $G[u]$, which satisfies a Lipschitz condition with constant L can be approximated arbitrarily closely on any closed finite interval by a monotone analytic function which also satisfies a Lipschitz condition with constant L, we can draw the following conclusion.

THEOREM 10. *If the input function $G[u]$ satisfies a Lipschitz condition on the interval*

$0 \leq u \leq 1 + \epsilon$, for any ϵ greater than zero, then the associated surface temperature function, $y(t)$, is non-decreasing for all t greater than zero.

This theorem is a simple consequence of Theorems 8 and 9, and the proof will be left to the reader. We shall merely indicate here one way in which any input function, $G[u]$, can be approximated on $[0, 1 + \epsilon]$ by an analytic input function. Define $H[u, \lambda]$ as follows

$$H[u, \lambda] = \frac{\lambda}{\pi} \int_{-\infty}^{\infty} \frac{G_1[u + \xi]}{\xi^2 + \lambda^2}\, d\xi = \frac{\lambda}{\pi} \int_{-\infty}^{\infty} \frac{G_1[x]}{(x - u)^2 + \lambda^2}\, dx$$

$$= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} G_1[u + \lambda \tan \theta]\, d\theta,$$

$$G_1[u] = \begin{cases} G[1 + \epsilon] & \text{when} \quad u > 1 + \epsilon, \\ G[u] & \text{when} \quad 0 \leq u \leq 1 + \epsilon, \\ G[0] & \text{when} \quad u < 0. \end{cases}$$

It is easily verified that $H[u, \lambda]$ is continuous in λ, that $\lim_{\lambda \to 0} H[u, \lambda] = G_1[u]$ and hence that the convergence is uniform on the closed interval $[0, 1 + \epsilon]$. By examining the difference $H[u_2, \lambda] - H[u_1, \lambda]$ it is evident that the monotony of G_1 on $[0, 1 + \epsilon]$ implies that of H, and it is obvious that if G_1 satisfies a Lipschitz condition with constant L, the same is true of H. Furthermore, $H[u, \lambda]$ is analytic in u.

THEOREM 11. *If the input function $G[u]$ satisfies a Lipschitz condition on an interval $[0, 1 + \epsilon]$ for any ϵ greater than zero, and if $y(t)$ is the corresponding surface temperature function, then $y(t) < 1$ for all $t \geq 0$, and $\lim_{t \to \infty} y(t) = 1$.*

Assume that $y(t_1) = 1$. Then since by Theorem 10 $y(t)$ is non-decreasing, and since $0 < y(t) \leq 1$ for all $t > 0$, it follows that $y(t) \equiv 1$ for all $t \geq t_1$. Let t_2 be any value of t greater than t_1. Then

$$y(t_1) = \int_0^{t_1} \frac{G[y(\tau)]}{\pi^{1/2}(t_1 - \tau)^{1/2}}\, d\tau = 1,$$

$$y(t_2) = \int_0^{t_1} \frac{G[y(\tau)]}{\pi^{1/2}(t_2 - \tau)^{1/2}}\, d\tau + \int_{t_1}^{t_2} \frac{G[y(\tau)]}{\pi^{1/2}(t - \tau)^{1/2}}\, d\tau.$$

But since $y(t) \equiv 1$ for $t_1 \leq t \leq t_2$, it follows from Hypothesis B that $G[y(\tau)] = 0$ for $t_1 \leq \tau \leq t_2$. Therefore the last equation reduces to

$$y(t_2) = \int_0^{t_1} \frac{G[y(\tau)]}{\pi^{1/2}(t_2 - \tau)^{1/2}}\, d\tau < \int_0^{t_1} \frac{G[y(\tau)]}{\pi^{1/2}(t_1 - \tau)^{1/2}}\, d\tau = 1,$$

since both integrands are positive for $0 \leq t \leq t_1$ and $t_2 > t_1$. This last conclusion that $y(t_2) < 1$ contradicts the earlier implication that $y(t) \equiv 1$ for all $t \geq t_1$. Hence our assumption that $y(t_1) = 1$ is impossible and we have that $y(t) < 1$ for all $t \geq 0$.

In order to prove that $\lim_{t \to \infty} y(t) = 1$, assume that this is false. Then by the non-decreasing nature of $y(t)$ we infer that there exists some number $\delta > 0$ such that $y(t) <$

$1 - \delta$ for all $t > 0$. By Hypotheses A, B, and C this means that $G[y(t)] > G[1 - \delta] > 0$ for all $t > 0$. This means that

$$y(t) > \int_0^t \frac{G[1 - \delta]}{\pi^{1/2}(t - \tau)^{1/2}} \, d\tau = \frac{G[1 - \delta]}{\Gamma(3/2)} \, t^{1/2}.$$

But the right-hand side of the above inequality can be made arbitrarily large by taking t sufficiently large. This contradicts the fact that $y(t) < 1$ for all $t > 0$. Hence δ does not exist and therefore $\lim_{t\to\infty} y(t) = 1$. This completes the proof of Theorem 11.

In view of this last result it is not hard to see how the hypotheses in Theorems 6, 7, 8, 10, and 11 can be somewhat weakened. Let $G[u]$ be any input function which satisfies a Lipschitz condition on the closed unit interval [0, 1]. Clearly, $G[u]$ can be continued outside this interval as an input function in such a way as to satisfy a Lipchitz condition in the large. One such extension is shown below:

$$G_1[u] = \begin{cases} 1 - u & \text{when} \quad u \geq 1, \\ G[u] & \text{when} \quad 0 \leq u < 1, \\ G[0] - u & \text{when} \quad u < 0. \end{cases}$$

By Theorem 6 there is a unique surface temperature function, $y(t)$, corresponding to G_1, and $y(t)$ is independent of the behavior of G_1 outside the unit interval. Since, by Theorem 11, $y(t) < 1$ for all $t \geq 0$, the uniqueness proof in Theorem 6 can be carried thru without regard to the behavior of G_1 outside the unit interval. Hence, the Lipschitz condition assumed in Theorems 6, 10, and 11 need hold only on the unit interval. Similarly, it is sufficient in Theorems 7 and 8 to require that G be analytic on the open interval (0, 1) and satisfy a Lipschitz condition on the closed interval.

4. Summary. Recalling that the input function, $G[U(0, t)]$ is simply the product of the film transfer factor, $f[U(0, t)]$ and the term $[1 - U(0, t)]/k$ we can summarize as follows the results which we have obtained:

CONCLUSION 1. For any film transfer factor of physical significance, the heat transfer problem stated in Sec. 2 always has at least one solution, $U(x, t)$, for all $x \geq 0$, $t \geq 0$. It can be constructed in the following way

$$U(x, t) = \int_0^t \frac{G[U(0, \tau)]}{\pi^{1/2}(t - \tau)^{1/2}} \exp \frac{-x^2}{4(t - \tau)} \, d\tau \tag{8}$$

from the surface temperature function $U(0, t)$ which must be a bounded solution of the following nonlinear integral equation

$$U(0, t) = \int_0^t \frac{G[U(0, \tau)]}{\pi^{1/2}(t - \tau)^{1/2}} \, d\tau. \tag{6}$$

Equation (6) always has at least one continuous bounded solution which satisfies the inequalities $0 < U(0, t) < 1$ for all $t > 0$, and having the property that $U(0, 0) = 0$. From these inequalities it follows by applying the mean value theorem to equation (8) that $0 < U(x, t) < 1$ for all $x \geq 0$, $t > 0$.

CONCLUSION 2. If $f[U(0, t)]$ satisfies a Lipschitz condition on the closed unit interval, then we can add to Conclusion 1 that

a) $U(0, t)$, and therefore $U(x, t)$, is unique.
b) $U(0, t)$ is non-decreasing for all $t > 0$.
c) $U(0, t) < 1$ for all $t \geq 0$.
d) $\lim_{t\to\infty} U(0, t) = 1$.

Conclusion 3. If the film transfer factor is analytic on the open unit interval and satisfies a Lipschitz condition on the closed unit interval, then we can add to Conclusion 2 that $U(0, t)$ is analytic and monotone increasing for all $t > 0$.

Conclusion 4. If the film transfer factor satisfies a Lipschitz condition on the closed unit interval, then $U(0, t)$ can be approximated uniformly, arbitrarily closely in the large by the method of successive approximations defined in formula (14). Since the successive approximations lie alternately above and below $U(0, t)$, an upper bound for the error in the nth approximating function is simply the difference $| U_n(0, t) - U_{n-1}(0, t) |$

As a final remark we point out that the methods developed here in the treatment of the surface temperature equation,

$$y(t) = \int_0^t \frac{G[y(\tau)]}{[\pi(t - \tau)]^{1/2}} d\tau \qquad (6)$$

are also applicable to a much more general equation,

$$y(t) = \int_0^t K(t, \tau)G[y(\tau)] \, d\tau \qquad (33)$$

in which the kernel, $K(t, \tau)$, need only be positive definite and satisfy appropriate integrability conditions. The theory for the more general equation (33) will be developed in a later paper.

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A NOTE ON THE EXISTENCE OF A SOLUTION TO A PROBLEM OF STEFAN*

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When certain metals are heated slowly, the temperature rises until it reaches a critical temperature at which the structure of the metal changes from one crystalline form to another. As for example, iron changes from α to β crystals at 1643°F. Accompanying this change of crystalline form is a latent heat of recrystallization. In order to study the process we investigate the associated mathematical problem, which requires the solution of a partial differential equation in a region with an undetermined boundary. Our analysis establishes the existence and uniqueness of the solution. In a previous paper[2] this problem is treated from the point of view of computing the solution.

Suppose a metal slab having two infinite parallel faces is brought uniformly to the critical temperature and then heated by a uniform source covering the front face while an insulator covers the back face. Under these conditions, the new crystals are first formed at the front face, and the interface between the new and old crystals travels from the front face to the back face. Mathematically the problem can be stated as follows, where $u = 0$ is taken as the critical temperature: Find the temperature, $u = u(x, t)$, and the curve, $x = x(t)$, which satisfy the following conditions

$$u_t = \alpha^2 u_{xx} \qquad \text{for } 0 < x < x(t) \tag{1}$$

$$u = 0 \qquad \text{for } x = x(t) \tag{2}$$

$$-Ax^{\cdot}(t) = u_x[x(t), t] \qquad \text{where } A > 0 \tag{3}$$

$$x(0) = 0 \tag{4}$$

$$u_x(0, t) = -g \tag{5}$$

where g is a constant >0.
In this notation

$$u_t = \partial u/\partial t, \qquad u_{xx} = \partial^2 u/\partial x^2, \qquad x^{\cdot}(t) = dx(t)/dt,$$

α^2 is the coefficient of thermal diffusivity, $A = \rho H/k$ where ρ is the density of the metal, H is the latent heat of recrystallization, and k is the coefficient of thermal conductivity.

We simplify the notation by introducing new variables as follows:

$$\begin{aligned} v(y, \tau) &= u(x, t)/A\alpha^2 \\ y &= gx/A\alpha^2 \\ \tau &= g^2 t/A^2\alpha^2; \end{aligned} \tag{6}$$

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[2]G. W. Evans II, E. Isaacson, and J. K. L. Mac Donald, *Stefan-like problems*, to be published in the Q. Appl. Math. **8**, 312-319 (1950).

and, then, by renaming v by u, τ by t, and y by x, the mathematical statement of the problem is:

Find $u = u(x, t)$ and $x = x(t)$ where the temperature $u(x, t)$ satisfies the following equation

$$u_{xx} = u_t \qquad \text{for } 0 < x < x(t) \tag{7}$$

with the boundary conditions

$$u = 0 \qquad \text{for } x = x(t) \tag{8}$$

$$x^{\cdot}(t) = -u_x[x(t), t] \tag{9}$$

$$x(0) = 0 \tag{10}$$

$$u_x(0, t) = -1. \tag{11}$$

In this discussion we will use a theorem of Dr. Louis Nirenberg[3] on the parabolic equation. For the requirements of this paper, a restricted statement of this theorem is given below as our principal lemma.

Lemma 1: Let R be a simply connected region in the x,t-plane where $0 \leq t \leq T$ with a part of the boundary of R being $t = T$ and the remaining part of the boundary being given locally by a curve $x = \beta(t)$. Furthermore, let $u(x, t)$ be a continuous and bounded solution of the heat conduction equation having continuous derivatives satisfying $u_{xx} = u_t$ in the interior of R, and let the solution be continuable into the region for which $t > T$. If $u(x, t)$ assumes its maximum or minimum in R, say at a point (ξ, τ), other than at a point of the boundary $x = \beta(t)$, then u is a constant in the subregion described as follows: the subregion consists of all points of R which may be reached by a continuous curve $t = f(s)$, $x = g(s)$, where $t = f(s)$ is a monotonic nonincreasing function of s, starting from any point in R that lies on the line $t = \tau$.

Furthermore, we assume the following two lemmas[4]:

Lemma 2: There exists a bounded solution, u, of $u_{xx} = u_t$ with bounded continuous first derivatives in the interior of the region $0 \leq t \leq T$, $0 \leq x \leq X(t)$ where $X(t) \geq 0$ is a curve with $X(0) = 0$, and u assumes the following boundary values: $u_x(0, t) = -1$ and $u[X(t), t] = 0$.

Lemma 3: There exists a bounded solution, u, of $u_{xx} = u_t$ with bounded continuous first derivatives in the interior of the region $0 \leq t \leq T$, $0 \leq x \leq X(t)$ where $X(t) \geq 0$ is a curve with $X(0) = 0$, and u assumes the following boundary values: $u_x(0, t) = 0$ and $u[X(t), t] = u(t) \geq 0$.

We now establish the existence of the solution $x = x(t)$, $u = u(x, t)$ of the problem given in (7)-(11). The proof consists in applying an iteration scheme to the equation

$$x(t) = t - \int_0^{x(t)} u(x, t)\, dx. \tag{12}$$

[3]This theorem is still to be published; but a similar theorem which is not as general, but which would be satisfactory for this paper, was proved by Mauro Picone (*Sul problema della propagazione del calore in un mezzo privo di frontiera, conducttore, isotropo e omogeneo*, Math. Ann. **101** (1929)).

[4]The problems of Lemmas 2 and 3 are of the type which were considered in a more general way by W. Sternberg (*Uber die Gleichung der Warmeleitung*, Math. Ann. **101** (1929)).

This equation is derived by evaluating

$$\int_0^t \int_0^{x(t)} (u_{xx} - u_t)\, dx\, dt = 0$$

with the boundary conditions (8), (9), (10), and (11). Our procedure determines $x(t)$ and $u(x, t)$ in the following way: let

$$x_n(t) = t - \int_0^{x_{n-1}(t)} u_{n-1}(x, t)\, dx \tag{13}$$

with $u_{n-1}(x, t)$ the solution of Lemma 2 for $X(t) = x_{n-1}(t)$. This procedure has been chosen because, after it has been shown that $u(x, t) = \operatorname{Lim}_{n\to\infty} [u_{n-1}(x, t)]$ is the solution of Lemma 2 with $X(t) = x(t) = \operatorname{Lim}_{n\to\infty} [x_n(t)]$, we may differentiate Eq. (12) with respect to t and find that

$$u_x[x(t), t] = -x^{\cdot}(t)$$

which is the boundary condition (9) of our problem not contained in Lemma 2.

The $x_n(t)$ for $n = 0, 1, 2, 3, \cdots$ are monotonic non-decreasing functions of t, i.e.

$$x_n^{\cdot}(t) \geq 0.$$

This may be seen by differentiating Eq. (13) with respect to t giving

$$x_n^{\cdot}(t) = -u_{n-1_x}[x_{n-1}(t), t].$$

And, it remains to show that $u_{n-1_x}[x_{n-1}(t), t] \leq 0$. By Lemma 1, $u_{n-1}(x, t)$ must have its maximum and minimum value along $x = 0$ since $u_{n-1}(x, t) \equiv C$ cannot satisfy the condition $u_{n-1_x}(0, t) = -1$. Furthermore, one can show $0 \leq u_{n-1}(0, t) < M$ where M is the upperbound of $u_{n-1}(x, t)$ of Lemma 2; and since $u_{n-1}[x_{n-1}(t), t] = 0$, then $u_{n-1_x}[x_{n-1}(t), t] \leq 0$.

To show that $0 \leq u_n(x, t) \leq M$ for $n = 0, 1, 2, 3, \cdots$, form $v_n(x, t) = u_n(x, t) + x$ where u_n satisfies the conditions of Lemma 2. $v_n(x, t)$, then, satisfies the equation

$$v_{n_{xx}}(x, t) = v_{n_t}(x, t) \tag{14}$$

with the boundary conditions

$$v_{n_x}(0, t) = 0 \qquad \text{and} \qquad v_n[x_n(t), t] = x_n(t)$$

and is a solution of Lemma 3. Since $v_{n_x}(0, t) = 0$, we reflect the solution about $x = 0$ giving the boundary conditions:

$$v_n[x_n(t), t] = x_n(t) \qquad \text{and} \qquad v_n[-x_n(t), t] = x_n(t).$$

Applying Lemma 1 to $v_n(x, t)$ in the region between the two curves $x = -x_n(t)$ and $x = x_n(t)$, we see that $v_n(x, t)$ must assume its maximum and minimum along $x = x_n(t)$, i.e., $v_n(x, t) \not\equiv C$ since $v_n[x_n(t), t] = x_n(t) \neq C$. For any given $t = \tau$

$$0 \leq v_n(x, t) \leq \text{maximum } [x_n(t)] \qquad \text{for} \qquad 0 \leq t \leq \tau$$

or

$$-x \leq u_n(x, t) \leq \text{maximum } [x_n(t)] - x \qquad \text{for} \qquad 0 \leq t \leq \tau.$$

But, $u_n(x, t)$ is defined only for $x \geq 0$ and assumes its maximum and minimum along $x = 0$, therefore

$$0 \leq u_n(x, t) \leq \text{maximum } [x_n(t)] - x \qquad \text{for} \qquad 0 \leq t \leq \tau.$$

Using, now, the monotonity of $x_n(t)$, we have

$$0 \leq u_n(x, t) \leq x_n(t) - x. \tag{15}$$

To show that the $\text{Lim}_{n \to \infty} [x_n(t)]$ exists, it is sufficient to show that

$$| x_{n+m}(t) - x_n(t) | \to 0 \qquad \text{as } n \to \infty$$

uniformly in t for sufficiently small t, say all $t < T_0$. For this, we choose $x_0(t) = t$, then

$$x_1(t) = t - \int_0^t u_0(x, t)\, dx$$

and

$$x_1(t) - x_0(t) = - \int_0^t u_0(x, t)\, dx. \tag{16}$$

From Eq. (16) and the inequality (15), $x_1(t)$ is seen to lie to the left of $x_0(t) = t$ in the x,t-plane where the x-axis is taken in the horizontal direction. Similarly,

$$x_2(t) - x_1(t) = - \int_0^{x_1(t)} [u_1(x, t) - u_0(x, t)]\, dx + \int_{x_1(t)}^t u_0(x, t)\, dx \geq 0 \tag{17}$$

if we can show $u_1(x, t) - u_0(x, t) \leq 0$. Let

$$v_1(x, t) = u_0(x, t) - u_1(x, t),$$

then $v_1(x, t)$ satisfies the differential equation

$$v_{1_{xx}}(x, t) = v_{1_t}(x, t)$$

with the boundary conditions:

$$v_{1_x}(0, t) = 0 \qquad \text{and} \qquad v_1[x_1(t), t] = u_0[x_1(t), t] \geq 0.$$

By Lemma 3 a solution exists to this problem; and since $v_{1_x}(0, t) = 0$ we can reflect the solution about $x = 0$ thus making $v_1(x, t)$ satisfy the alternate boundary conditions:

$$v_1[x_1(t), t] = u_0[x_1(t), t] \geq 0 \qquad \text{and} \qquad v_1[-x_1(t), t] = u_0[x_1(t), t] \geq 0.$$

By applying Lemma 1 with these boundary conditions, $v_1(x, t)$ must assume its maximum and minimum along $x_1(t)$; and since $v_1(x, t) \not\equiv C$, $v_1(x, t) \geq 0$. By the expression (17), $x_2(t)$ must lie to the right of $x_1(t)$; and by considering

$$x_2(t) - x_0(t) = - \int_0^{x_1(t)} u_1(x, t)\, dx,$$

$x_2(t)$ must also lie to the left of $x_0(t) = t$. In fact, by replacing $x_2(t)$ by $x_n(t)$ in the preceding argument, all $x_n(t)$, for $n = 2, 3, 4, \cdots$, must lie between $x_0(t)$ and $x_1(t)$. By continuing this process

$$x_3(t) - x_2(t) = - \int_0^{x_1(t)} [u_2(x, t) - u_1(x, t)]\, dx - \int_{x_1(t)}^{x_2(t)} u_2(x, t)\, dx \leq 0 \tag{18}$$

since

$$v_2(x, t) = u_2(x, t) - u_1(x, t) \geq 0;$$

and

$$x_3(t) - x_1(t) = -\int_0^{x_2(t)} [u_2(x, t) - u_0(x, t)]\, dx + \int_{x_2(t)}^{t} u_0(x, t)\, dx \geq 0. \tag{19}$$

The expressions (18) and (19) show that $x_3(t)$ lies to the left of $x_2(t)$ and to the right of $x_1(t)$. By further continuing the process it is seen that

$$x_{2k-1}(t) \leq x_{2k}(t) \leq x_{2(k-1)}(t)$$

and

$$x_{2k-1}(t) \leq x_{2k+1}(t) \leq x_{2k}(t)$$

for $k = 1, 2, 3, \cdots$. That is, each curve $x_i(t)$, $i = 2, 3, 4, \cdots$, lies in the region between the preceding two curves, $x_{i-1}(t)$ and $x_{i-2}(t)$.

To establish the existence of the $\operatorname{Lim}_{n\to\infty} [x_n(t)]$, consider the difference, as defined by Eq. (13) of

$$x_{n+1}(t) - x_n(t) = -\int_0^{x_n(t)} u_n(x, t)\, dx + \int_0^{x_{n-1}(t)} u_{n-1}(x, t)\, dx. \tag{20}$$

If $x_n(t)$ lies to the right of $x_{n-1}(t)$, then Eq. (20) becomes

$$x_{n+1}(t) - x_n(t) = \int_0^{x_{n-1}(t)} [u_{n-1}(x, t) - u_n(x, t)]\, dx + \int_{x_n(t)}^{x_{n-1}(t)} u_n(x, t)\, dx, \tag{21}$$

and if $x_n(t)$ lies to the left of $x_{n-1}(t)$, Eq. (20) becomes

$$x_{n+1}(t) - x_n(t) = \int_0^{x_n(t)} [u_{n-1}(x, t) - u_n(x, t)]\, dx + \int_{x_n(t)}^{x_{n-1}(t)} u_{n-1}(x, t)\, dx. \tag{22}$$

To estimate the absolute value of the difference in Eq. (21), choose a time, say $t = T$, and by using the expression (15), Eq. (21) becomes

$$\begin{aligned} | x_{n+1}(T) - x_n(T) | = \Bigg| \int_0^{x_{n-1}(T)} & [u_{n-1}(x, T) - u_n(x, T)]\, dx \\ & + \int_{x_n(T)}^{x_{n-1}(T)} u_n(x, T)\, dx \Bigg|. \end{aligned} \tag{23}$$

Since $u_n(x, T) < T$ (for $0 \leq u_n(x, T) \leq u_n(0, T) \leq T$), by substituting T for $u_n(x, T)$ in the second term of the right hand side of Eq. (23), the left hand side satisfies the following expression

$$\begin{aligned} | x_{n+1}(T) - x_n(T) | \leq \int_0^{x_{n-1}(T)} & | [u_n(x, T) - u_{n-1}(x, T)] | \cdot | dx | \\ & + T \, | x_n(T) - x_{n-1}(T) |. \end{aligned} \tag{24}$$

To estimate the integral in the inequality (24), form

$$v(x, t) = u_n(x, t) - u_{n-1}(x, t)$$

which satisfies

$$v_{xx}(x, t) = v_t(x, t)$$

with the boundary conditions

$$v_x(0, t) = u_{n_x}(0, t) - u_{n-1_x}(0, t) = 0$$

and

$$v[x_{n-1}(t), t] = u_n[x_{n-1}(t), t] \geq 0.$$

By Lemma 3 a solution exists and since $v_x(0, t) = 0$, reflect the solution about $x = 0$, thus making $v(x, t)$ satisfy the alternate boundary conditions

$$v[x_{n-1}(t), t] = u_n[x_{n-1}(t), t] \geq 0$$

$$v[-x_{n-1}(t), t] = u_n[x_{n-1}(t), t] \geq 0.$$

Now, by applying Lemma 1, $v(x, t)$ must assume its maximum along $x_{n-1}(t)$ since $v(x, t) \not\equiv C$. But, by the inequality (15), along $x = x_{n-1}(t)$,

$$u_n[x_{n-1}(t), t] - u_{n-1}[x_{n-1}(t), t] = u_n[x_{n-1}(t), t] \leq x_n(t) - x_{n-1}(t).$$

Therefore

$$| u_n(x, T) - u_{n-1}(x, T) | \leq | x_n(T) - x_{n-1}(T) |. \tag{25}$$

The inequality (24) now becomes

$$| x_{n+1}(T) - x_n(T) | \leq | x_n(T) - x_{n-1}(T) | \int_0^{x_{n-1}(T)} | dx | + T | x_n(T) - x_{n-1}(T) |$$

$$\leq 2T | x_n(T) - x_{n-1}(T) |$$

since $x_{n-1}(T) < T$. Now choose $T_0 = \frac{1}{4}$, then for $t = T \leq \frac{1}{4}$

$$| x_{n+1}(t) - x_n(t) | \leq \tfrac{1}{2} | x_n(t) - x_{n-1}(t) |. \tag{26}$$

The result (26) may be deduced in a similar manner from Eq. (22). The inequality (26) establishes the existence of the $\text{Lim}_{n\to\infty} [x_n(t)] = x(t)$ for $t \leq \frac{1}{4}$.

Since for $t \leq \frac{1}{4}$ one has already shown that $| x_{n+1}(t) - x_n(t) |$ may be made less than ϵ for $n > N(\epsilon)$, and since by the expression (25)

$$| u_n(x, t) - u_{n-1}(x, t) | \leq | x_n(t) - x_{n-1}(t) |,$$

the $\text{Lim}_{n\to\infty} [u_n(x, t)] = u^*(x, t)$ exists for $t \leq \frac{1}{4}$.

Since it is clear that the limits of the iterations satisfy the integral equation (12), it remains to show that $u^*(x, t)$ satisfies the differential equation (1) with the boundary conditions of Lemma 2. To do this let $u(x, t)$ be defined as the solution of Lemma 2 for $x = x(t)$ where $x(t) = \text{Lim}_{n\to\infty} [x_n(t)]$. Set

$$x^*(t) = t - \int_0^{x(t)} u(x, t)\, dx,$$

and form the difference

$$x^*(t) - x_n(t) = \int_0^{x_{n-1}(t)} [u_{n-1}(x, t) - u(x, t)]\, dx - \int_{x_{n-1}(t)}^{x(t)} u(x, t)\, dx. \tag{27}$$

Next, consider the limit of Eq. (27) as $n \to \infty$, but choose the sequence through which n varies so that $x_{n-1}(t)$ always lies to the left of $x(t)$. Then

$$\operatorname*{Lim}_{n\to\infty} [x^*(t) - x_n(t)] = \operatorname*{Lim}_{n\to\infty} \left\{ \int_0^{x_{n-1}(t)} [u_{n-1}(x, t) - u(x, t)]\, dx \right\};$$

so set

$$-v_{n-1}(x, t) = u_{n-1}(x, t) - u(x, t)$$

which satisfies the equation

$$v_{n-1_{xx}}(x, t) = v_{n-1_t}(x, t)$$

with the boundary conditions

$$v_{n-1_x}(0, t) = 0 \qquad \text{and} \qquad v_{n-1}[x_{n-1}(t), t] = u[x_{n-1}(t), t] = f(t).$$

Therefore $v_{n-1}(x, t)$ satisfies Lemma 3. Since $v_{n-1_x}(0, t) = 0$, reflect $v_{n-1}(x, t)$ about $x = 0$, then $v_{n-1}(x, t)$ still satisfies the same differential equation with the alternate boundary conditions

$$v_{n-1}[x_{n-1}(t), t] = f(t) \qquad \text{and} \qquad v_{n-1}[-x_{n-1}(t), t] = f(t).$$

By Lemma 1 either $v_{n-1}(x, t) \equiv 0$ since $v_{n-1}(0, 0) = 0$ which is what we desire, or the maximum or minimum of $v_{n-1}(x, t)$ lies on $x = x_{n-1}(t)$. If the second condition is true, let n tend to infinity in the prescribed manner, then

$$\operatorname*{Lim}_{n\to\infty} \{v_{n-1}[x_{n-1}(t), t]\} = \operatorname*{Lim}_{n\to\infty} \{-u[x_{n-1}(t), t]\} = -u[x(t), t] = 0.$$

This implies that $x^*(t) = x(t)$ which implies that $u^*(x, t) = u(x, t)$ or that $u^*(x, t)$ satisfies the differential equation (1).

In verifying the solution of the problem, one must show that $dx(t)/dt$ exists where $x(t) = \operatorname{Lim}_{n\to\infty} [x_n(t)]$. This is easily shown by the definition of a derivative and with the aid of Eq. (12):

$$\frac{dx(t)}{dt} = \operatorname*{Lim}_{\Delta t\to 0} \left[\frac{x(t + \Delta t) - x(t)}{\Delta t} \right] \tag{28}$$

or

$$\frac{x(t + \Delta t) - x(t)}{t} = \frac{t + \Delta t - t}{t} - \frac{\int_0^{x(t+\Delta t)} u(x, t + \Delta t)\, dx - \int_0^{x(t)} u(x, t)\, dx}{\Delta t}$$

$$= 1 - \int_0^{x(t)} \left[\frac{x(u, t + \Delta t) - u(x, t)}{\Delta t} \right] dx - \int_{x(t)}^{x(t+\Delta t)} \frac{u(x, t + \Delta t)}{\Delta t}\, dx \tag{28'}$$

$$= 1 - \int_0^{x(t)} u_t[x, t + \theta(x)\Delta t]\, dx - \frac{u[\phi(\Delta t), t + \Delta t]}{\Delta t} \int_{x(t)}^{x(t+\Delta t)} dx$$

where $0 \leq \theta(x) \leq 1$ and $x(t) \leq \phi(\Delta t) \leq x(t + \Delta t)$. $u[\phi(\Delta t), t + \Delta t]$ is a mean of the values u takes as it varies from $u[x(t), t + \Delta t]$ to $u[x(t + \Delta t), t + \Delta t]$. Using the last expression above, the left hand side of Eq. (28′) may be written as

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} \frac{1 - \int_0^{x(t)} u_t[x, t + \theta(x)\Delta t]\, dx}{1 + u[\phi(\Delta t), t + \Delta t]} . \tag{29}$$

The integral on the left hand side of Eq. (29) may be evaluated as follows:

$$\int_0^{x(t)} u_t[x, t + \theta(x)\Delta t]\, dx = \int_0^{x(t)} u_{xx}[x, t + \theta(x)\Delta t]\, dx. \tag{30}$$

Since $u_{xx}(x, t)$ is continuous and bounded for all finite $t \geq 0$ and $0 \leq x \leq x(t)$ we may write Eq. (30) as

$$\int_0^{x(t)} u_t[x, t + \theta(x)\Delta t]\, dx = \int_0^{x(t)} [u_{xx}(x, t) + \epsilon]\, dx = u_x[x(t), t] - u_x(0, t) + \epsilon x(t).$$

By using the continuity of $u_{xx}(x, t)$ and by considering a finite region $0 \leq x \leq x(t)$, $0 \leq t < T^*$, the term $\epsilon x(t)$ may be neglected since $x(t) < T^*$. Since $u(x, t)$ is continuous in the same region as $u_{xx}(x, t)$, the

$$\lim_{\Delta t \to 0} \{u[\phi(\Delta t), t + \Delta t]\} = u[x(t), t] = 0.$$

Therefore, we may let $\Delta t \to 0$ on the right hand side of Eq. (29) to obtain

$$\frac{dx(t)}{dt} = -u_x[x(t), t]. \tag{31}$$

To show the uniqueness of $x = x(t) = \lim_{n \to \infty} [x_n(t)]$, assume two solutions of our problem, $U(x, t)$ producing the curve $x = X(t)$ and $V(x, t)$ producing the curve $x = Y(t)$. If $X(t) \equiv Y(t)$ then it is already known that $U(x, t) \equiv V(x, t)$;[5] so assume $X(t) > Y(t)$ for $t > 0$. From Eq. (12) we have

$$X(t) = t - \int_0^{X(t)} U(x, t)\, dx$$

$$Y(t) = t - \int_0^{Y(t)} V(x, t)\, dx.$$

Form the difference

$$X(t) - Y(t) = \int_0^{Y(t)} [V(x, t) - U(x, t)]\, dx - \int_{Y(t)}^{X(t)} U(x, t)\, dx. \tag{32}$$

Since by Eq. (15), $U(x, t) \geq 0$, set $W(x, t) = V(x, t) - U(x, t)$ which satisfies the equation

$$W_{xx}(x, t) = W_t(x, t) \tag{33}$$

with the boundary conditions

$$W_x(0, t) = 0 \quad \text{and} \quad W[Y(t), t] = -U[Y(t), t] \leq 0.$$

[5]L. Bieberbach, *Differentialgleichungen*, Dover Publications, New York, 1944, pp. 391-392.

Since $W_x(0, t) = 0$ reflect this solution about $x = 0$ so that $W(x, t)$ still satisfies Eq. (33) but with the alternate boundary conditions

$$W[-Y(t), t] = -U[Y(t), t] \leq 0$$

and

$$W[Y(t), t] = -U[Y(t), t] \leq 0 \quad .$$

Applying Lemma 1 to $W(x, t)$ we see that since it cannot satisfy $W(x, t) \equiv C$, its maximum value is at (0, 0) and

$$W(x, t) \leq 0.$$

By forming the difference.

$$X(t) - Y(t) = \int_0^{Y(t)} W(x, t)\, dx - \int_{Y(t)}^{X(t)} U(x, t)\, dx \leq 0, \tag{34}$$

Eq. (34) contradicts our assumption that $X(t) > Y(t)$. Since X can be replaced by Y and U by V in the above argument to give a contradiction on the assumption that $Y(t) > X(t)$, then $X(t) = Y(t)$. The uniqueness of our solution is shown under the assumption that $X(t)$ and $Y(t)$ do not intersect infinitely often as $t \to 0$.

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—NOTES—

THE CONSERVATION OF SYSTEMS IN PHASE SPACE*

By ERNEST IKENBERRY (*Louisiana State University*)

1. Introduction. Important progress in the development of a statistical theory of the transport phenomena in liquids, based on the application of Gibbs' principle of "conservation of density-in-phase,"[1] has been made in recent papers by Kirkwood,[2] by Born and Green,[3] and by Jaffé.[4] Kirkwood and Born and Green have derived the Maxwell-Boltzmann integro-differential equation and have developed a general statistical mechanical theory of transport processes, by application of the principle of continuity in phase space. In their method of treatment of this principle, Born and Green defined a set of multiform distribution functions and obtained a generalized equation of motion referring to a cluster of h molecules. This generalized equation reduces to the equation of motion of ordinary hydrodynamics when $h = 1$. Born and Green also obtained expressions for the coefficients of thermal conductivity and of viscosity, but have not published any numerical results.

Using a method analogous to that introduced by Boltzmann[5] in the kinetic theory of gases, Jaffé[4] obtained a solution for the distribution function in the form of a multiple power series proceeding according to powers of the kinetic energy and resultant moments. Previously determined[6] potential functions were used to calculate theoretical values for the coefficients of thermal conductivity and of viscosity of ten liquids. The numerical results obtained agree reasonably well with the observed values, in most instances.

In each of the three methods of treatment the consideration of boundary conditions has been almost completely avoided. Kirkwood and Born and Green have suggested making the distribution function vanish at the boundaries, and Jaffé limited his considerations to the neighborhood of a particle in the interior of the liquid. However, it would seem that, since the coefficients of heat conductivity and of viscosity are defined in terms of the transfer of thermal energy and of momentum through boundary surfaces, the theory of these dissipative processes cannot be considered as sufficiently well developed without the further consideration of boundary conditions. In our treatment of the principle of conservation of density-in-phase, the boundary conditions to

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[1] J. Willard Gibbs, *Collected Works*, Longman, Green & Co., New York, 1931, Vol. II, Part I, "Elementary Principles in Statistical Mechanics," Ch. 1.

[2] J. G. Kirkwood, J. Chem. Phys. **14**, 180 (1946).

[3] M. Born and H. S. Green, Nature **159**, 251 (1947); Proc. Roy. Soc. London (A) **188**, 10 (1946), and subsequent papers.

[4] G. Jaffé, Phys. Rev. **69**, 688 (1946), **75**, 184 (1949).

[5] L. Boltzmann, *Vorlesungen über Gastheorie*, Johann A. Barth, 1896, 1. Theil, "Theorie der Gase mit einatomigen Molekülen, deren Dimensionen gegen die mittlere Weglänge verschwinden," p. 184.

[6] G. Jaffé, Phys. Rev. **62**, 463 (1942).

be applied refer to the boundaries of the region of phase space occupied by the systems of a virtual ensemble, rather than to the boundary condition problems as treated in hydrodynamics. The formulation of boundary conditions in phase space will be preceded by the statement of what may reasonably be assumed regarding boundary conditions in real space, in the most general case of a fluid contained in a finite volume V with moving boundaries.

A fluid system bounded by the walls of a container consists always of the same particles, but consideration of such a system is complicated by the presence of a boundary region of finite thickness where the particles of the fluid are in interaction with the molecules of the material comprising the walls. We may simplify the boundary conditions, in some respects, by considering a system which is entirely surrounded by a larger system of the same kind. Due to diffusion, the boundaries of the real system would become ill-defined if we were to require that the real system is to always consist of the same particles. We may, however, define the motion of the boundaries of the real system in such a way that, during any reasonably short interval of time, there is no mass current (see Eq. (2.12)) through the boundaries. During such an interval of time, there is no *net* diffusion of particles through the boundaries of the real system.

The analogy between the hydrodynamical fluid in real space and the imaginary fluid consisting of systems of the virtual ensemble in phase space may be extended by conceiving of a diffusion of the systems of the virtual ensemble. In order that the boundaries of the virtual ensemble should not become ill-defined, it is sufficient that the boundaries of the virtual ensemble should be defined in terms of the previously defined boundaries of the real system in real space. The boundary condition to be applied is that, during any reasonably short interval of time, there is no *net* diffusion of systems through the boundaries of the region of phase space occupied by the systems of the virtual ensemble. During such an interval of time, both the number of particles in the real system and the number of systems in the virtual ensemble remain constant.

In the present paper, boundary conditions of this very general nature will be applied in the derivation of an equation of transport analogous to the equations of transfer derived by Maxwell[7] in his kinetic theory of gases. The ordinary equation of continuity, the hydrodynamical equations of motion, and the equations of thermal and of total energy will then be obtained as special cases of this general transport equation.

2. Preliminary definitions. We are considering a system of N identical particles, each of mass m. The state of the system at any instant is determined by the values of the $3N$ position coordinates (x_a, y_a, z_a) and of the corresponding momenta $(m(dx_a/dt)$, $m\,(dy_a/dt)$, $m\,(dz_a/dt))$ of the N particles. Let

$$q_{3a-2} = x_a, \qquad q_{3a-1} = y_a, \qquad q_{3a} = z_a, \qquad a = 1, 2, \cdots N, \tag{2.1}$$

and

$$p_i = m\frac{dq_i}{dt}, \qquad\qquad i = 1, 2, \cdots 3N. \tag{2.2}$$

Then the $6N$ values of the q_i and the p_i at any instant determine a point in a $6N$ dimensional phase space. This point is representative of the instantaneous state of the

[7] J. C. Maxwell, Phil. Trans. **157**, 1 (1886), or *Collected Works*, II, p. 26; J. H. Jeans, *The dynamical theory of gases*, The Univ. Press, Cambridge, 1925, Ch. IX.

system, and the progress of the system in time is represented by the motion of its representative point along a path in phase space. Following a method developed by Gibbs,[1] we consider a virtual ensemble of **N** such systems, each consisting of N identical particles, all of the systems being subject to the same internal and external forces, but distinguished in that they are distributed among the various states accessible to the actual system. In general, the ranges of the position coordinates q_i are determined by the physical volume occupied by the particles of the actual system. The ranges of the momentum coordinates p_i are limited by the maximum kinetic energy which the real system may have, but to avoid mathematical difficulties, we will follow the customary procedure by taking all real values for the ranges of the momentum coordinates

Letting

$$d\Omega_0 = \prod_{i=1}^{3N} dq_i \, dp_i \tag{2.3}$$

be the element of extension in phase space, we write

$$d\mathbf{N} = f(q, p, t) \, d\Omega_0 \tag{2.4}$$

for the number of systems in $d\Omega_0$ at time t. The distribution function f, which gives the instantaneous density of systems in phase space at the point (q, p), is in general a function of the $6N + 1$ variables q, p, t. This function is subject to certain conditions analogous to the conditions first formulated by Hilbert[8] for the distribution function of a gas in position-velocity space. In a strict sense, f is not a continuous function of the $6N + 1$ variables. Nevertheless, it is assumed that the density of systems in phase space is so great that f can be approximated to a sufficient degree of accuracy by a function which is continuous and has continuous derivatives with respect to the phase coordinates and the time. Since a density is of necessity a positive quantity, the distribution function must not become negative within the ranges of these $6N + 1$ variables. The distribution function must also be symmetrical in the phase coordinates of any two particles, inasmuch as any one of the N identical particles may be chosen as the representative particle. It is further assumed that the distribution function vanishes sufficiently rapidly, for large values of the momentum coordinates, to assure the existence of all integrals of the form $\int f\phi \, d\Omega_0$, where $\phi(q, p, t)$ is any polynomial in the momentum coordinates with coefficients continuous in the position coordinates and in the time.

The average value, $\langle\phi\rangle$, of a function ϕ of the phase coordinates and of the time, for a specified instant of time and for a specified position of a representative particle, say particle No. 1, is defined by

$$\langle\phi\rangle \int f \, d\Omega_1 = \int f\phi \, d\Omega_1 , \tag{2.5}$$

where

$$d\Omega_1 = \prod_{i=4}^{3N} dq_i \prod_{i=1}^{3N} dp_i \tag{2.6}$$

is the element of extension in the subspace Ω_1 of Ω_0. We note that Ω_1 is the section of Ω_0 obtained by assigning specified values to q_1, q_2, and q_3. Selection of particle No. 1 as representative particle means that, henceforth, (q_1, q_2, q_3) may also be considered

[8] D. Hilbert, Math. Ann. **72**, 562 (1912).

as the coordinates of a point in real space and need not be distinguished from (x_1, y_1, z_1). From (2.5) and (2.6) it may be understood that $\langle\phi\rangle$ is, in general, a function of the four variables q_1, q_2, q_3, and t even when ϕ itself does not depend explicitly on all four of these variables. This arises from the, in general, complicated manner in which the distribution function depends upon the position coordinates and on the time.

Letting

$$d\tau = dq_1\, dq_2\, dq_3 \tag{2.7}$$

be the element of extension in real space, and defining $d\mathbf{N}_1$ by

$$d\mathbf{N}_1 = d\tau \int f\, d\Omega_1\,, \tag{2.8}$$

we may make the transformation to real space in the customary manner by writing

$$\frac{dN}{N} = \frac{d\mathbf{N}_1}{\mathbf{N}}\,, \tag{2.9}$$

where dN is the number of particles contained in $d\tau$ at a specified instant of time. Hence, by (2.8) and (2.9), the density n of particles in real space is given by

$$n = \frac{dN}{d\tau} = \frac{N}{\mathbf{N}} \int f\, d\Omega_1\,. \tag{2.10}$$

We note from (2.5) and (2.10) that

$$\int f\phi\, d\Omega_1 = \frac{N}{\mathbf{N}}\, n\langle\phi\rangle \tag{2.11}$$

Letting $i, j = 1, 2,$ or 3, the components mu_i of the mass current, the components S_{ij} of the strain tensor, the temperature T, and the components H_i of the heat current, at a specified instant of time and at a specified point (q_1, q_2, q_3), are defined by

$$mu_i = \langle p_i\rangle, \tag{2.12}$$

$$S_{ij} = \frac{n}{m}\langle p_i' p_j'\rangle \tag{2.13}$$

$$3nkT = S_{11} + S_{22} + S_{33}\,, \tag{2.14}$$

and

$$H_i = \frac{n}{2m^2}\langle p_i'\{(p_1')^2 + (p_2')^2 + (p_3')^2\}\rangle \tag{2.15}$$

respectively, where $p_i' = p_i - \langle p_i\rangle$.

3. The conservation of systems in phase space. If we define the operator D by

$$D = \frac{\partial}{\partial t} + \sum_{i=1}^{3N}\left\{\frac{dq_i}{dt}\frac{\partial}{\partial q_i} + \frac{dp_i}{dt}\frac{\partial}{\partial p_i}\right\}, \tag{3.1}$$

the hydrodynamical equation of continuity in phase space assumes the form[9]

$$Df + f\sum_{i=1}^{3N}\left\{\frac{\partial}{\partial q_i}\left(\frac{dq_i}{dt}\right) + \frac{\partial}{\partial p_i}\left(\frac{dp_i}{dt}\right)\right\} = 0. \tag{3.2}$$

[9] Jeans, *op. cit.*, p. 71.

The phase coordinates are connected by Hamilton's equations of motion,

$$\frac{dq_i}{dt} = \frac{\partial \epsilon}{\partial p_i}, \qquad \frac{dp_i}{dt} = \frac{-\partial \epsilon}{\partial q_i} + F_i\,, \qquad i = 1, 2, \cdots 3N, \tag{3.3}$$

where ϵ is the Hamiltonian function for the system and the F_i are the components of non-conservative forces. Hereafter we shall, in most cases, designate the dp_i/dt by R_i. The forces whose components are the R_i may be partly of internal, partly of external origin. In case each force component R_i is independent of the corresponding momentum coordinate p_i, the $3N$ terms in the summation in (3.2) vanish and (3.2) leads to Gibbs' principle of conservation of density-in-phase, which is valid for non-conservative as well as for conservative systems, so long as $\partial R_i/\partial p_i = 0$.

When there is no diffusion of systems in phase space, the vanishing of the summation in (3.2) is equivalent to Gibbs' principle of extension-in-phase. This principle, together with Gibbs' principle of conservation of density-in-phase, would then imply conservation of the total number of systems in the virtual ensemble. However, when there is diffusion of systems in phase space, the motion in phase space of the boundary surface S_0 of Ω_0 is not determined by (3.3), and, in order to obtain conservation of the total number of systems in the virtual ensemble, we must supplement (3.2) by the boundary condition that there is no net flow of systems through the boundaries of the region of phase space available to the systems of the virtual ensemble. We proceed to give the mathematical formulation of this boundary condition, for the general case in which the boundary surface S_0 of Ω_0 is in motion in any arbitrary manner.

For complete generality, we consider a function $\psi = f\phi$ where ϕ is any polynomial in the momentum coordinates with coefficients which may be functions of the position coordinates and of the time. Then $\partial/\partial t \int \psi\, d\Omega_0$ is defined as the limit, as $\Delta t \to 0$, of the difference quotient of $\int \psi\, d\Omega_0$ evaluated at times t and $t + \Delta t$:

$$\frac{\partial}{\partial t}\int_{\Omega_0} \psi(q, p, t)\, d\Omega_0 = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\{\int_{\Omega'_0} \psi(q, p, t + \Delta t)\, d\Omega_0 - \int_{\Omega_0} \psi(q, p, t)\, d\Omega_0\right\}, \tag{3.4}$$

where Ω_0 and Ω_0' are the regions of phase space occupied by the systems of the virtual ensemble at times t and $t + \Delta t$, respectively. Assuming that $\psi(q, p, t + \Delta t)$ may, for sufficiently small values of Δt, be expanded as a Maclaurin's series in Δt, we obtain, to terms of the first order in Δt,

$$\int_{\Omega'_0} \psi(q, p, t + \Delta t)\, d\Omega_0 = \int_{\Omega'_0} \psi(q, p, t)\, d\Omega_0 + \Delta t \int_{\Omega_0} \frac{\partial \psi(q, p, t)}{\partial t}\, d\Omega_0\,. \tag{3.5}$$

Hence (3.4) becomes

$$\frac{\partial}{\partial t}\int_{\Omega_0} \psi\, d\Omega_0 = \int_{\Omega_0} \frac{\partial \psi}{\partial t}\, d\Omega_0 + \lim_{\Delta t \to 0} \frac{1}{\Delta t}\int_{\Omega'_0 - \Omega_0} \psi\, d\Omega_0\,, \tag{3.6}$$

where $\Omega_0' - \Omega_0$ is the region of phase space through which the boundary surface S_0 of Ω_0 moves during time Δt. Letting $V_0(n)$ be the component of the velocity of motion of the element dS_0 of the boundary surface S_0 of Ω_0, normal to dS_0, we may write

$$d\,\Omega_0 = V_0(n)\, d\, S_0\, \Delta t, \tag{3.7}$$

where $d\Omega_0$ is the region of phase space through which dS_0 moves during time dt. Hence (3.6) becomes

$$\frac{\partial}{\partial t}\int_{\Omega_0}\psi\,d\Omega_0=\int_{\Omega_0}\frac{\partial\psi}{\partial t}\,d\Omega_0+\int_{S_0}\psi V_0(n)\,dS_0\,. \tag{3.8}$$

We now let $v_0(n)$ be the component of the velocity of motion of a representative point at the surface element dS_0 of the surface S_0 of Ω_0 , normal to dS_0 . By Gauss' theorem for the transformation of a surface integral into a volume integral we may write

$$\int_{S_0}\psi v_0(n)\,dS_0=\int_{\Omega_0}\sum_{i=1}^{3N}\left\{\frac{\partial}{\partial q_i}\left(\psi\,\frac{dq_i}{dt}\right)+\frac{\partial}{dp_i}\left(\psi\,\frac{dp_i}{dt}\right)\right\}d\Omega_0\,. \tag{3.9}$$

Upon combining (3.8) and (3.9), we obtain, by means of (3.1) and (3.4),

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\Omega_0}\psi\,d\Omega_0=\int_{\Omega_0}\left[D\psi+\psi\sum_{i=1}^{3N}\left\{\frac{\partial}{\partial q_i}\left(\frac{dq_i}{dt}\right)+\frac{\partial}{\partial p_i}\left(\frac{dp_i}{dt}\right)\right\}\right]d\Omega_0\\+\int_{S_0}\psi\{V_0(n)-v_0(n)\}\,dS_0\,.\end{aligned} \tag{3.10}$$

With $\psi = f$, we see that (3.10) states that the rate of increase of the total number of systems in the virtual ensemble is equal to the rate at which systems are produced in Ω_0 , augmented by the net rate of flow of systems into Ω_0 through its boundaries. In order to obtain conservation of the total number of systems in the virtual ensemble, we shall apply not only (3.2), but also the boundary condition that there is no net flow of systems through the boundaries of the region of phase space available to the systems of the virtual ensemble, so that the surface integral in (3.10) vanishes. We note that the surface integral in (3.10) may vanish without the integrand itself vanishing identically, that is, without $V_0(n) = v_0(n)$. For equality of these two generalized velocity components would imply that there is no transport of any quantity ϕ through the boundaries of the system. As a concrete example, we may consider the case of a fluid contained within rigid boundaries, with heat being applied at the boundaries. Then, to the extent that the walls are rigid, the boundaries of Ω_0 must also be considered as fixed, so that $V_0(n) = 0$. But, as there is transfer of heat into the system through the walls, the particles must, on the average, rebound from the walls with increased kinetic energy. This may be pictured in phase space by imagining that a system of the virtual ensemble flows out through the boundary of Ω_0 at the instant a particle strikes the rigid wall and another system flows in through the boundary when the particle rebounds from the wall, an infinitesimally short time later.

4. The equation of transport. An equation of transport, including the general case of moving boundaries, may be derived directly from (3.2) by the imposition of a boundary condition in the $6N - 3$ dimensional subspace Ω_1 of Ω_0 , analogous to the boundary condition previously imposed in Ω_0 . Let $V_1(n)$ be the component of the velocity of a point on the element dS_1 of the boundary surface S_1 of Ω_1 , normal to dS_1 , and let $v_1(n)$ be the component of the velocity of a representative point, in the subspace Ω_1 , at the moment that point is on or passes through dS_1 , normal to dS_1 . By exactly the same procedure as previously used, we obtain, corresponding to (3.8),

$$\frac{\partial}{\partial t}\int_{\Omega_1}\psi\, d\Omega_1 = \int_{\Omega_1}\frac{\partial\psi}{\partial t}\, d\Omega_1 + \int_{S_1}\psi V_1(n)\, dS_1\ . \tag{4.1}$$

Since Ω_1 is the space of all $3N$ momentum coordinates but of only $3N - 3$ position coordinates, application of Gauss' theorem for the transformation of a surface integral into a volume integral gives

$$\int_{S_1}\psi v_1(n)\, dS_1 = \int_{\Omega_1}\left\{\sum_{i=4}^{3N}\frac{\partial}{\partial q_i}\left(\psi\frac{dq_i}{dt}\right) + \sum_{i=1}^{3N}\frac{\partial}{\partial p_i}\left(\psi\frac{dp_i}{dt}\right)\right\} d\Omega_1\ . \tag{4.2}$$

We now supply on both sides of (4.2) the three terms necessary to make the indices in both summations to run from 1 to $3N$, combine the result with (4.1), replace ψ by $f\phi$, and apply (3.2) to obtain

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\Omega_1} f\phi\, d\Omega_1 &+ \sum_{i=1}^{3}\frac{\partial}{\partial q_i}\int f\frac{p_i}{m}\phi\, d\Omega_1\\ &= \int_{\Omega_1} f\, D\phi\, d\Omega_1 + \int_{S_1} f\phi\{V_1(n) - v_1(n)\}\, dS_1\ .\end{aligned} \tag{4.3}$$

When each member of (4.3) is multiplied by $N\ \Delta\tau/\mathbf{N}$, where $\Delta\tau$ is any fixed element of volume in real space, the first term on the left hand side of the resulting equation then represents the time rate of increase of the total amount of ϕ in $\Delta\tau$, and the summation represents the net rate of flow of the quantity ϕ out of $\Delta\tau$, through its boundaries. The first term on the right hand side, involving $D\phi$, then represents the rate of increase of the total amount of ϕ in $\Delta\tau$, due to the explicit dependence of ϕ on the phase coordinates and on the time. However, $V_1(n)$ and $v_1(n)$ are generalized velocity components in the imaginary subspace Ω_1 of phase space, and the surface integral over dS_1, in the right hand side of (4.3), then represents the rate at which the quantity ϕ is transported into $\Delta\tau$, but not through its boundaries. The real physical space is a 3-dimensional subspace of phase space, and there are paths in generalized coordinate space which lead from the exterior to the interior of the closed volume in real space, yet which do not pass through the boundaries of the real volume. As the transport of any physical quantity from the exterior to the interior of a closed volume, by a path not passing through its boundaries, is experimentally unobservable, we require that the surface integral in (4.3) shall vanish. We thus obtain a first form of the equation of transport:

$$\frac{\partial}{\partial t}\int f\phi\, d\Omega_1 + \sum_{i=1}^{3}\frac{\partial}{\partial q_i}\int f\frac{p_i}{m}\phi\, d\Omega_1 = \int f(D\phi)\, d\Omega_1\ , \tag{4.4}$$

which is valid for any quantity ϕ whose value for the entire system is equal to the sum of its values for the individual particles, and for which the integrals involved exist. A second form of the equation of transport (4.4) is readily obtained by the aid of (2.11):

$$\frac{\partial}{\partial t}\langle n\phi\rangle + \frac{1}{m}\sum_{i=1}^{3}\frac{\partial}{\partial q_i}\langle np_i\phi\rangle = n\langle D\phi\rangle, \tag{4.5}$$

since both N and $\mathbf{N}$ are constant. The right hand side of (4.4), and of (4.5), remains to be developed, for any particular ϕ, by means of (3.1).

5. The equation of continuity, the equations of motion, and the energy equations. Letting $\phi = 1$, we obtain from (4.5), by means of (2.12),

$$\frac{\partial n}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial q_i} (nu_i) = 0, \tag{5.1}$$

which is the equation of continuity. Elimination of $\partial n/\partial t$ from (4.5) and (5.1) results in a third form of the equation of transport:

$$\frac{d}{dt} \langle \phi \rangle + \frac{1}{nm} \sum_{i=1}^{3} \frac{\partial}{\partial q_i} \langle np'_i \phi \rangle = \langle D\phi \rangle, \tag{5.2}$$

where the operator d/dt is defined by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{3} u_i \frac{\partial}{\partial q_i} \cdot \tag{5.3}$$

Letting $\phi = p_j$, $D\phi = R_j$, in (5.2), we obtain the hydrodynamical equations of motion,

$$\frac{d}{dt} (mu_j) + \frac{1}{n} \sum_{i=1}^{3} \frac{\partial S_{ij}}{\partial q_i} = \langle R_j \rangle, \qquad j = 1, 2, \text{ or } 3, \tag{5.4}$$

the components S_{ij} of the strain tensor having been defined in (2.13). We note that the mean force acting on the representative particle may be non-conservative even when the forces acting on the individual particles are conservative. Conservative internal forces may tend to increase or decrease the disordered motion of the particles composing the system, and this is interpreted, from the macroscopic point of view, as the effect of non-conservative forces acting on the parcels of the fluid, as in the appearance of tangential surface forces when there are velocity gradients.

In order to obtain the equation of thermal energy we may let $\phi = (1/2m) \sum_{j=1}^{3} (p'_j)^2$ in (5.2). By the aid of (2.12) to (2.15) and of (3.1) we obtain

$$\frac{d}{dt}\left(\frac{3}{2} kT\right) + \frac{1}{n} \sum_{i=1}^{3} \frac{\partial H_i}{\partial q_i} + \frac{1}{n} \sum_{i,j=1}^{3} S_{ij} \frac{\partial u_j}{\partial q_i} = \frac{1}{m} \sum_{i=1}^{3} \langle R_i p'_i \rangle. \tag{5.5}$$

It is seen that external conservative forces, which depend at most on the position coordinates of the particle on which they act, and possibly on the time, make no contribution to the right hand side of (5.5). In the case of an ideal gas, whose internal energy is entirely kinetic, the right hand member of (5.5) vanishes. The resulting equation may then be compared with equation (13) of Enskog's dissertation.[10] The terms in the right hand member of (5.5) become negligible in the case of a moderately dense gas for which the internal forces decay sufficiently rapidly with distance.

The equation of total energy may be obtained from (5.2) by letting $\phi = \epsilon_1 = \varphi_1^{(e)} + \varphi_1^{(i)} + (1/2m)(p_1^2 + p_2^2 + p_3^2)$, where $\varphi_1^{(e)}$ and $\varphi_1^{(i)}$ are the time independent external and internal potentials of the representative particle. The external potential $\varphi_1^{(e)}$ will be assumed to be a function of the position coordinates q_1, q_2, and q_3 of the representative particle only. Furthermore, only binary attractions and repulsions will be considered, so that the total potential of the entire system may be expressed as the sum of the po-

[10]D. Enskog, *Kinetische Theorie der Vorgänge in mässig verdünnten Gasen*, Inaugural Dissertation, Upsala, 1917, p. 18.

tentials of the individual particles. Upon substituting ϵ_1 for ϕ into (5.2), multiplying the resulting equation by $n\,d\tau$, and integrating over $d\tau$ at a constant time, we obtain, by the aid of (2.3), (2.6), (2.7), (2.11) to (2.15) and (3.1) to (3.3), and by considerations of symmetry,*

$$\begin{aligned} \frac{d}{dt}\int n\langle\epsilon_1\rangle\,d\tau + \int \sum_{i=1}^{3} \frac{\partial}{\partial q_i}\left(H_i + \sum_{j=1}^{3} S_{ij}u_j\right) d\tau + \int \sum_{i=1}^{3} \frac{\partial}{\partial q_i}\left\{\frac{n}{m}\,\langle p_i'\varphi_1^{(i)}\rangle\right\} d\tau& \\ = \int \sum_{i=1}^{3} \frac{n}{m}\,\langle p_i F_i\rangle\,d\tau,& \end{aligned} \tag{5.6}$$

where

$$\langle\epsilon_1\rangle = \frac{1}{2}\,m(u_1^2 + u_2^2 + u_3^2) + \varphi_1^{(e)} + \langle\varphi_1^{(i)}\rangle + \frac{3}{2}\,kT. \tag{5.7}$$

Upon transforming the second and third volume integrals on the left hand side of (5.6) into surface integrals, we see that this equation may be interpreted as stating that the time rate of increase of total energy of the system is equal to the rate at which the non-conservative body forces F_i and surface forces S_{ij} do work on the system, augmented by the rates of transport of thermal energy of disordered motion and of internal potential energy through its boundaries. The importance of this latter portion of the energy flux, arising from the strong intermolecular forces in the case of a liquid, has been pointed out by Born and Green.[11] This flux is not contained in the expression (2.15) for the components of the heat current.

*In the equation obtained by letting $\phi = \epsilon_1$, in (5.2), we interchange the phase coordinates of the representative particle with those of each of the other $N - 1$ particles in turn, and add the resulting equations. Then, by considerations of symmetry and by the aid of (2.3), (2.6), (2.7) and (2.11), $\int n\langle\epsilon_a\rangle d\tau = (N/\mathbf{N}) \int f\epsilon_a\,d\Omega_0 = (N/\mathbf{N}) \int f\epsilon_1\,d\Omega_0 = \int n\langle\epsilon_1\rangle\,d\tau$. A similar method of treatment applies to the other terms on the left hand side of the combined equation. The right hand side of this equation becomes $(N/\mathbf{N}) \int f(D\epsilon)\,d\Omega_0 = (N/\mathbf{N}) \int f\, \Sigma_{i=1}^{3N} (p_iF_i/m)\,d\Omega_0 = N \int \Sigma_{i=1}^{3} (m/n)\langle p_iF_i\rangle\,d\tau$, by the aid also of (3.2) and (3.3).

[11] M. Born and H. S. Green, Proc. Roy. Soc. London (A) **190**, 455 (1947).

NOTE ON THE HAMEL-SYNGE THEOREM*

By F. H. van den DUNGEN (*Université Libre de Bruxelles*)

The theorem given by Synge[1] for a plane motion of a compressible viscous fluid is easily extended to a three dimensional motion.

Consider a compressible viscous fluid which moves inside a fixed closed surface B, on which the velocity vanishes. Our theorem is: A velocity $\mathbf{v}(x, y, z)$ is consistent with the foregoing boundary condition if and only if

$$\int_V (\mathbf{A}\cdot\operatorname{curl}\mathbf{v} - f\operatorname{div}\mathbf{v})\,dx\,dy\,dz = 0, \tag{1}$$

*Received June 16, 1950.

[1] Q. Appl. Math. 8, 107-108 (1950)

where f is an arbitrary harmonic function and $\mathbf{A}$ a conjugate harmonic vector such that

$$\text{grad } f + \text{curl } \mathbf{A} = 0. \tag{2}$$

Proof. We have

$$\int_V (\mathbf{A} \cdot \text{curl } \mathbf{v} - \mathbf{v} \cdot \text{curl } \mathbf{A}) \, dx \, dy \, dz = -\iint_S \mathbf{A} \cdot (\mathbf{v} \times \mathbf{n}) \, dS$$

(here $\mathbf{n}$ is the normal unit vector) and

$$\int_V (f \text{ div } \mathbf{v} + \mathbf{v} \cdot \text{grad } f) \, dx \, dy \, dz = \iint_B f\mathbf{v} \cdot \mathbf{n} \, dS,$$

and therefore it follows that

$$\iint_B (\mathbf{A} \cdot (\mathbf{v} \times \mathbf{n}) + f\mathbf{v} \cdot \mathbf{n}) \, dS \tag{3}$$

vanishes if and only if Eq. (1) is satisfied. However, (3) will vanish for arbitrary harmonic f only if a continuous $\mathbf{v}$ vanishes everywhere on the surface B, and thus (1) is necessary and sufficient for the satisfaction of the viscous boundary condition.

NOTE ON ITERATIONS WITH CONVERGENCE OF HIGHER DEGREE*

By HASKELL B. CURRY (*Pennsylvania State College*)

In a recent paper[1] E. Bodewig has derived a general expression for a function $F(x)$ such that the sequence

$$x_{n+1} = F(x_n) \tag{A}$$

converges in a degree at least m in the neighborhood of every root of a polynomial $f(x)$, provided the latter has only simple roots. A part of this argument can be simplified, while another part can be made somewhat more natural.

In regard to the first point, the operator P used by Bodewig is the same as d/dy where $y = f(x)$. Further, since his r is $1/f'$, we have $r = dx/dy = Px$. Hence the identity between Bodewig's formula (14a) and the Euler inverse Taylor expansion (15) follows immediately; it is not necessary to use the recurrence formula for the higher derivatives of an inverse function.

In regard to the second point, if we have a sequence of functions $F_1(x)$, $F_2(x)$, $\cdots$ such that the sequence (A) converges in the degree at least m for $F = F_m$, then any $F(x)$ giving rise to a sequence converging at least in the degree m must be of the form

$$F = F_m + g_m f^m.$$

*Received August 29, 1950.

[1]E. Bodewig, *On types of convergence and on the behavior of approximations in the neighborhood of a multiple root of an equation*, Q. Appl. Math. **7**, 325-333 (1949).

Now F_{m+1} is itself such an F. Hence, by induction, we have, for suitable g_k,

$$F_m = x + g_1 f + g_2 f^2 + \cdots + g_{m-1} f^{m-1}.$$

This suggests a change of variable to $y = f(x)$ (which is possible since $f'(X) \neq 0$). If $x = u(y)$ is the inverse function, and

$$\Phi_m(y) = F_m[u(y)], \qquad \psi_k(y) = g_k[u(y)],$$

then

$$\Phi_m = u + \psi_1 y + \psi_2 y^2 + \cdots + \psi_{m-1} y^{m-1}. \tag{B}$$

The conditions which must be satisfied by Φ_m are

$$\Phi(0) = X,$$

$$\Phi'(0) = \Phi''(0) = \cdots = \Phi^{m-1}(0) = 0.$$

The first of these is automatically satisfied.

Now a function $\Phi_m(y)$ satisfying these conditions is given immediately by the inverse Taylor expansion of $X = u(y - y)$. In fact, if we set $\Phi_m(y)$ equal to the sum of the first m terms of this expansion, viz.:

$$\Phi_m(y) = \sum_{k=0}^{m-1} \frac{(-y)^k}{k!} u^{(k)}(y),$$

then

$$X = \Phi_m(y) + \frac{(-y)^m}{m!} u^{(m)}(\eta),$$

and hence

$$\Phi_m(y) = X - \frac{(-y)^m}{m!} u^{(m)}(\eta)$$

satisfies the above conditions.

This method avoids the necessity of slapping down Bodewig's formula (14) or of motivating it by tedious experimenting with small values of m.

BOUNDARIES FOR THE LIMIT CYCLE OF VAN DER POL'S EQUATION*

By R. GOMORY and D. E. RICHMOND (*Williams College*)

1. Introduction. In non-linear mechanics much interest centers on the Van der Pol (VDP) equation

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0 \tag{1}$$

*Received Sept. 11, 1950.

or its equivalent in the phase plane

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -x + \mu(1 - x^2)y. \tag{2}$$

It is well known that (2) possesses a unique trajectory which represents a limit cycle in the sense of Poincaré. Using a different plane, La Salle[1] has located this limit cycle between two boundary curves in a very ingenious manner which however seems artificial and difficult to motivate. The present paper sets forth a simple and natural method for constructing outer and inner boundaries. The method admits of unlimited improvement but even its simplest application gives results superior to La Salle's in that the limit cycle is localized somewhat more sharply.

In the phase plane all trajectories other than the limit cycle spiral into it from the inside or the outside. The curves are described clockwise with increasing t. We use these facts to enclose the limit cycle between an outer boundary B_0 and an inner boundary B_i .

Introducing $r^2 = x^2 + y^2$, we transform (2) to

$$r\frac{dr}{dt} = \mu(1 - x^2)y^2, \qquad \frac{dx}{dt} = y. \tag{3}$$

Eliminating t, one obtains for the trajectories

$$r\frac{dr}{dx} = \mu(1 - x^2)y. \tag{4}$$

Since the field of (4) is symmetrical in the origin, it is sufficient to discuss solutions in the upper half-plane ($y \geqq 0$).

If C is a curve, $r = F(x)$, $y \geqq 0$, which intersects the x-axis only at $(-a, 0)$ and $(a, 0)$ and if at every point the value of dr/dx for C is *greater than* or equal to that of the VDP solution through that point, all VDP curves intersect C from above to below. Then C together with its image in the origin forms an *outer* boundary B_0 .

The construction of an *inner* boundary B_i requires the substitution of *less than* for *greater than* in the above statement.

2. The outer boundary. To construct an outer boundary B_0 , write (4) in the form

$$r\frac{dr}{dx} = \mu(1 - x^2)(r^2 - x^2)^{1/2}, \qquad y \geq 0, \tag{4'}$$

On an x-interval within which $1 - x^2$ is positive, replace x^2 under the radical by $(x^2)_{\min}$, its least value on the interval. The curves defined by the solutions of

$$r\frac{dr}{dx} = \mu(1 - x^2)[r^2 - (x^2)_{\min}]^{1/2}, \qquad y \geq 0, \tag{5}$$

have at every point of the interval a value of dr/dx greater than or equal to that of the corresponding VDP curve.

Similarly, on an x-interval within which $1 - x^2$ is negative, replace x^2 under the radical by $(x^2)_{\max}$, its largest value on the interval. The curves defined by

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[1] J. La Salle, *Relaxation o scillations*, Q. Appl. Math. **7**, 1-19 (1949). If La Salle's co-ordinates are (x, u), the relation to ours is given by $y/\mu + u = x - (x^3/3)$. His t is μ times ours.

$$r\frac{dr}{dx} = \mu(1 - x^2)[r^2 - (x^2)_{\max}]^{1/2}, \qquad y \geq 0, \tag{6}$$

have at every point of the interval a value of dr/dx greater than or equal to that of the corresponding VDP curve.

It remains to join together solutions of (5) and (6), valid over different intervals, to generate a continuous curve which will serve as C, the portion of B_0 in the upper half-plane.

This boundary, at least for large μ, may be expected to lie rather close to the limit cycle since over much of the short x range, x^2 is small relative to r^2 so that the error made by the proposed substitution is not serious. In fact, by using a large number of intervals a very accurate outer boundary may be constructed. But even the simplest outer boundary, using three intervals, is surprisingly good. We proceed to the details for this case.

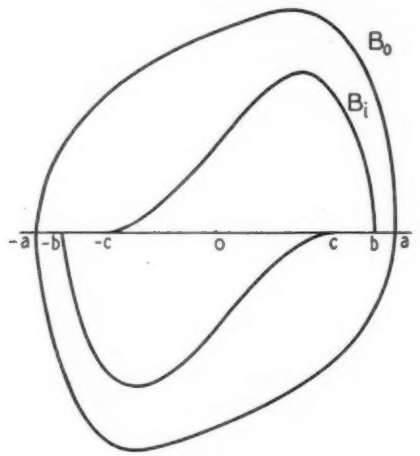

FIG. 1. Sketch of Boundaries.

Let C intersect the x-axis at $-a$ and a. The intervals to be used are $[-a, -1]$, $[-1, 1]$ and $[1, a]$. We start at $-a$ and work across to a in the upper half plane. The equations become

$$[-a, -1] \qquad r\frac{dr}{dx} = \mu(1 - x^2)(r^2 - a^2)^{1/2},$$

$$[-1, 1] \qquad r\frac{dr}{dx} = \mu(1 - x^2)(r^2 - 0^2)^{1/2},$$

$$[1, a] \qquad r\frac{dr}{dx} = \mu(1 - x^2)(r^2 - a^2)^{1/2}.$$

Each of these equations is separable and possesses a general and a singular solution. We use the singular solution $r = a$ for $[-a, -1]$ since it is the only solution through

$(-a, 0)$ which is real in the interval. For the other intervals, we use the general solutions which take the form

$$(r^2 - c^2)^{1/2} = \mu\left(x - \frac{x^3}{3}\right) + k.$$

It is easy to choose the constants k to secure continuity.

The principal interest is in the amplitude a which determines the size of the enclosure. To find a, we take definite integrals successively over the intervals $[-a, -1]$, $[-1, 1]$ and $[1, a]$, finding in each case the relation between r at the left side and r at the right side of the interval. The results are

$$\begin{aligned} r(-1) - a &= 0, \\ r(1) - r(-1) &= \frac{4\mu}{3}, \\ [r^2(1) - a^2]^{1/2} &= \frac{\mu}{3}(a^3 - 3a + 2). \end{aligned} \tag{7}$$

Combining

$$a^3 - 3a + 2 - 4\left(1 + \frac{3a}{2\mu}\right)^{1/2} = 0.$$

Thus even in the simplest case there is a slight improvement over La Salle's result, which in this form is

$$a^3 - 3a + 2 - 4\left(1 + \frac{3a}{4\mu}\right) = 0.$$

We quote some numerical results for $\mu = 3$. La Salle's boundary gives $a = 2.21^+$. Our result is $a = 2.18^-$. If seven intervals are used (joining at $-3^{1/2}$, -1.6, -1.4, -1, 1 and $3^{1/2}$), one obtains $a = 2.10^-$.

3. The inner boundary. To construct an *inner* boundary B_i, use

$$r\frac{dr}{dx} = \mu(1 - x^2)(r^2 - x^2)^{1/2}, \qquad y \geq 0, \tag{4'}$$

as before, but replace x^2 under the radical by $(x^2)_{\max}$ if $1 - x^2$ is positive within the x-interval, by $(x^2)_{\min}$ if $1 - x^2$ is negative there.

Using three intervals $[-c, -1]$, $[-1, 1]$ and $[1, b]$, we obtain the equation

$$r\frac{dr}{dx} = \mu(1 - x^2)(r^2 - 1)^{1/2},$$

applicable to all three intervals. One is tempted to set $c = b$ and reflect in the origin. There is however a difficulty. To the left of $x = -1$, dr/dx is negative but $-dr/dx \leqq 1$ for $x \geqq -c$. This condition restricts the choice of c.

Since $r(-dr/dx) = \mu(x^2 - 1)(r^2 - 1)^{1/2}$, $y \geqq 0$, and since on the x-axis, $r = c$, it follows that

$$c \geq \mu(c^2 - 1)^{3/2}.$$

The largest c corresponds to the equality. If

$$d = \mu(d^2 - 1)^{3/2}$$

defines d for a given μ, we therefore choose $c \leqq d$ and follow the curve along the upper half-plane to its right-hand intersection at b. Reflection of this curve in the origin leaves two gaps, one between $-b$ and $-c$ and one between c and b. If $b > c$ we may use segments of the x-axis to complete the inner boundary. Since the VDP curves are described clockwise with increasing t, these curves cross the added segments in the required direction.

The relation between $-c$ and b is given by integrating between these limits and is

$$(b^2 - 1)^{1/2} - (c^2 - 1)^{1/2} = \mu\left(b - \frac{b^3}{3} + c - \frac{c^3}{3}\right). \tag{9}$$

We know that $c \leqq d$. The largest inner boundary using three intervals arises from the choice $c = d$. However, for the sake of simplicity, we may obtain an inner boundary valid for all μ by choosing $c = 1$. Then (9) becomes

$$b^3 - 3b - 2 + \frac{3}{\mu}(b^2 - 1)^{1/2} = 0,$$

which is a slight improvement over La Salle's result

$$b^3 - 3b - 2 + \frac{3b}{\mu} = 0.$$

For $\mu = 3$, for example, La Salle's result is $b = 1.77$. Ours is $b = 1.81$ with $c = 1$ and $b = 1.87^+$ with $c = d = 1.248$. If additional points of division are placed at $-.5$, $.5$ and $3^{1/2}$, $b = 1.94$ is obtained.

4. Conclusion. In conclusion, it is clear that the simplicity of the calculations makes it relatively easy to obtain indefinitely better boundaries by increasing the number of intervals. Over these intervals, the form of the solution is (with one exception) always the same, different $(x)^2_{\max}$ and $(x)^2_{\min}$ being inserted. It is therefore sufficient to be armed with a table of square roots and cubes to find a or b by trial solution. With a moderate amount of work, it is fortunately possible to supplement the method of perturbations which is useful for $\mu \ll 1$ and the known results for $\mu \to \infty$ by giving a good account of the limit cycle in the intermediate range of μ.

It is also obvious that the method of this paper can be applied if $x^2 - 1$ in Eq. (1) is replaced by other suitable functions $f(x)$. Further generalizations are possible but will not be discussed.

A THIRD ORDER BOUNDARY VALUE PROBLEM ARISING IN AEROELASTIC WING THEORY*

By GEORGE SEIFERT** (*Cornell University*)

The differential equation arising in the problems of chordwise divergence or swept-forward wing bending divergence [1] is of the type

$$\frac{d^2}{dx^2} EI(x) \frac{dz}{dx} - \lambda c(x) z = 0 \tag{1a}$$

where λ is a parameter. The boundary conditions are $z(0) = z'(l) = (EI(x)z'(x))'_{x=l} = 0$. Since this represents a non self-adjoint boundary value problem, the ordinary Rayleigh-Ritz variational method for approximating its characteristic values is not applicable. Formal extensions of this variational approach have been suggested by Flax [1] and used by Cheng [2] but no mathematical evidence for their validity is at present available. In this connection, two questions have been raised [1].

(A) Under what conditions on $EI(x)$ and $c(x)$ are all the characteristic values of (1a) real?

(B) Under what conditions on $EI(x)$, $c(x)$ and $f(x)$ can the solution $z_\lambda(x)$ of

$$[EI(x)z'(x)]'' - \lambda c(x)z(x) = f(x), \qquad z(0) = z'(l) = (EIz')'_l = 0$$

be expanded in terms of the biorthogonal system of functions arising from the system (1a) and its adjoint?

Although this paper concerns itself chiefly with (A), it may be pointed out that the investigations of expansions in terms of characteristic functions of (1a) for $EI \equiv 1$, $c(x) \equiv -1$, $l = \pi$ carried out by L. E. Ward [3] indicate difficulties inherent in such irregular boundary-value expansion problems as (B).

Part I of this paper deals with the characteristic values of the system

$$u'''(x) + p(x)u'(x) + [q(x) + \lambda]u(x) = 0, \qquad u(0) = u'(0) = u''(1) = 0, \tag{1}$$

where p and q are real-valued functions analytic on $0 \leq x \leq 1$. It is found that:

(a) this system has an infinite number of real characteristic values (Theorem 1);

(b) if the upper bounds of $| p(t) |$ and $| q(t) |$, and the positive numbers $| p(1) |$, $\int_0^1 | p(t) | \, dt$, $\int_0^1 | r(t) | \, dt$, where $r(t) = q(t) - p'(t)$, are small enough, then all characteristic values of (1) are real (Theorem 3).

Just how small these positive quantities in (b) must be, in order that all the characteristic values of (1) be real, is not explicitly considered. For specific functions p and q, however, the details of the proofs of (a) and (b) enable one to answer this question by means of simple numerical computations.

Much of the method of proof and notation is similar to that used by Ward [4] in his study of the system:

$$u'''(x) + [\rho^3 + r(x)]u(x) = 0, \qquad u(0) = u'(0) = u(\pi) = 0.$$

Here $r(x)$ is of a special form which permits Ward to obtain an expansion theorem. It

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is hoped that Ward's approach will also suggest an expansion theorem associated with (1) and perhaps also provide an answer to (B).

Part II indicates a pair of transformations which takes the system

$$[f(t)y''(t)]' + \lambda g(t)y(t) = 0, \qquad y(0) = y'(0) = y''(1) = 0, \tag{2}$$

where $f(t) > 0$, $g(t) > 0$, are real and analytic on $0 \leq t \leq 1$, over into a system of the form of (1). Note that the adjoint of (2) is of the form of (1a), and consequently the characteristic values of (2), being identical with those of its adjoint, are of interest in aeroelastic wing theory.

PART I

We consider the system (1). Define

(a) $\delta_1(t) = e^{\omega_1 t} + e^{\omega_2 t} + e^{\omega_3 t}$

$\delta_2(t) = e^{\omega_1 t} - \omega_3 e^{\omega_2 t} - \omega_2 e^{\omega_3 t}$

$\delta_3(t) = e^{\omega_1 t} - \omega_2 e^{\omega_2 t} - \omega_3 e^{\omega_3 t}$

where

$$\omega_1 = -1, \qquad \omega_2 = e^{(\pi i)/3}, \qquad \omega_3 = e^{-(\pi i)/3};$$

(b) the complex number ρ by $\rho^3 = \lambda$, $|\arg \rho| \leq \pi/3$;

(c) the regions S_1 and S_2 of the ρ plane by $0 \leq \arg \rho \leq \pi/3$ and $-\pi/3 \leq \arg \rho \leq 0$ respectively.

Lemma 1. *A necessary and sufficient condition that $u(x, \rho)$ satisfy the equation in* (1) and $u(0, \rho) = u'(0, \rho) = 0$[1] *is that*

$$u(x, \rho) = \delta_3(\rho x) - (3\rho^2)^{-1} \int_0^x \{\delta_3[\rho(x - t)]r(t) - \rho\delta_2[\rho(x - t)]p(t)\}u(t, \rho)\, dt \tag{1.1}$$

where $r(t) = q(t) - p'(t)$.

The proof of this lemma is completely analogous to that of Theorem 1 of Ward's paper[2] and is omitted here.

Lemma 2. *A solution $u(x, \rho)$ of the equation in* (1) *such that $u(0, \rho) = u'(0, \rho) = 0$ is given by*

$$u(x, \rho) = e^{\omega_3 \rho x}[-\omega_3 - \omega_2 e^{(\omega_2 - \omega_3)\rho x} + z(x, \rho)] \tag{1.2}$$

where $|z(x, \rho)| \leq m$, m being independent of x and ρ, provided $\rho \epsilon S_1$ and $|\rho|$ sufficiently large.

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[1]Unless otherwise indicated, the prime will always denote differentiation with respect to the first variable.

[2]L. E. Ward, [4], p. 418.

Proof. Substituting (1.2) into (1.1) one obtains

$$e^{\omega_3 \rho x}[A(x, \rho) + z(x, \rho)]$$

$$= \delta_3(\rho x) - (3\rho^2)^{-1} \int_0^x \{\delta_3[\rho(x - t)]r(t) - \rho\delta_2[\rho(x - t)]p(t)\}[A(t, \rho) + z(t, \rho)]e^{\omega_3 \rho t}\, dt,$$

where $A(t, \rho) = -\omega_3 - \omega_2 \exp[(\omega_2 - \omega_3)\rho t]$. Hence

$$z(x, \rho) = e^{(\omega_1 - \omega_3)\rho x}$$

$$- (3\rho^2)^{-1} \int_0^x \{\delta_3[\rho(x - t)]r(t) - \rho\delta_2[\rho(x - t)]p(t)\}[A(t, \rho) + z(t, \rho)]e^{-\omega_3 \rho(x-t)}\, dt. \tag{1.3}$$

Now for fixed $\rho \epsilon S_1$ $| z(x, \rho) |$ attains its upper bound $m(\rho)$ on $0 \leq x \leq 1$; hence by (1.3)

$$m(\rho) \leq 1 + (3 \mid \rho \mid^2)^{-1} \int_0^1 \mid \delta_3[\rho(x - t)]A(t, \rho)e^{-\omega_3 \rho(x-t)}r(t) \mid dt$$

$$+ (3 \mid \rho \mid^2)^{-1} \int_0^1 \mid \delta_3[\rho(x - t)]z(t, \rho)e^{-\omega_3 \rho(x-t)}r(t) \mid dt$$

$$+ (3 \mid \rho \mid)^{-1} \int_0^1 \mid \delta_2[\rho(x - t)]A(t, \rho)e^{-\omega_3 \rho(x-t)}p(t) \mid dt$$

$$+ (3 \mid \rho \mid)^{-1} \int_0^1 \mid \delta_2[\rho(x - t)]z(t, \rho)e^{-\omega_3 \rho(x-t)}p(t) \mid dt$$

$$\leq 1 + m(\rho)\left(\frac{r}{\mid \rho \mid^2} + \frac{p}{\mid \rho \mid}\right) + 2\left(\frac{r}{\mid \rho \mid^2} + \frac{p}{\mid \rho \mid}\right)$$

where

$$r = \int_0^1 \mid r(t) \mid dt, \qquad p = \int_0^1 \mid p(t) \mid dt.$$

Hence

$$m(\rho) \leq \frac{1 + 2(r/\mid \rho \mid^2 + p/\mid \rho \mid)}{1 - (r/\mid \rho \mid^2 + p/\mid \rho \mid)}$$

for $| \rho |$ sufficiently large, from which the lemma follows.

Theorem 1. *There exist an infinite number of real characteristic values λ_n for the system* (1); *more precisely, there exists a real λ_k such that all λ_n with $R(\lambda_n) > \lambda_k$ are necessarily real.*[3]

Proof. From (1.1) by differentiation,

$$u''(x, \rho) = \rho^2 \delta_1(\rho x) - \frac{1}{3} \int_0^x \{\delta_1[\rho(x - t)]r(t) - \rho\delta_3[\rho(x - t)]p(t)\}u(t, \rho)\, dt$$

$$- p(x)u(x, \rho),$$

[3]If $z = x + iy$, then $R(z) = x$ defines the notation $R(z)$ which will be used throughout.

and the characteristic equation, $u''(1, \rho) = 0$, becomes

$$\rho^2\delta_1(\rho) - \frac{1}{3}\int_0^1 \{\delta_1[\rho(1-t)]r(t) - \rho\delta_3[\rho(1-t)]p(t)\}[A(t, \rho) + z(t, \rho)]e^{\omega_3\rho t}\, dt$$

$$- p(1)[A(1, \rho) + z(1, \rho)]e^{\omega_3\rho} = 0.$$

Define

$$E(\rho) = (3\rho^2)^{-1}\int_0^1 \{\delta_1[\rho(1-t)]r(t) - \rho\delta_3[\rho(1-t)]p(t)\}[A(t, \rho) + z(t, \rho)]e^{-\omega_3\rho(1-t)}\, dt$$

$$+ p(1)[A(1, \rho) + z(1, \rho)]\rho^{-2}.$$

It is easily verified that the zeros of $\delta_1(\rho)$ are the zeros of $\exp(-3\rho/2) + 2\cos[3^{1/2}\rho/2]$ and are all real for $\rho \in S_1$.

It will now be shown that $|\rho^2\delta_1(\rho)| > |\rho^2 e^{\omega_3\rho}E(\rho)|$ for ρ on any one of a set of contours C_n in the ρ plane which are trapezoids whose bases are the segments

$$R(\rho) = 2(n+1)\frac{\pi}{3^{1/2}}, \qquad |\arg \rho| \le \frac{\pi}{3};$$

$$R(\rho) = 2(n+2)\frac{\pi}{3^{1/2}}, \qquad |\arg \rho| \le \frac{\pi}{3};$$

$n = k, k+1, \cdots, k$ sufficiently large. It is sufficient to show this inequality for the upper half of the contour C_n; namely $C'_n = C_n \cap S_1$. This is true because $u(x, \rho)$ is real for real ρ and hence $u(x, \rho^*) = [u(x, \rho)]^*$, the star denoting the complex conjugate. Hence $u''(x, \rho^*) = [u''(x, \rho)]^*$, and consequently $\rho^2[\delta_1(\rho) - e^{\omega_3\rho}E(\rho)]$ takes on values on $C''_n = C_n \cap S_2$, the lower half of C_n, which are the complex conjugates of those taken on on C'_n, the upper half. Set $F(\rho) = \rho^2\delta_1(\rho)$, $G(\rho) = -\rho^2 e^{\omega_3\rho}E(\rho)$; then since $F(\rho^*) + G(\rho^*) = [F(\rho)]^* + [G(\rho)]^*$ and $F(\rho^*) = [F(\rho)]^*$, we have $G(\rho^*) = [G(\rho)]^*$, and hence $\rho^2 e^{\omega_3\rho}E(\rho)$ assumes values on C''_n which are the complex conjugates of those it takes on on C'_n. It clearly suffices, then, to show that

$$|e^{-\omega_3\rho}\delta_1(\rho)| > |E(\rho)| \qquad \text{for } \rho \text{ on } C'_n. \tag{1.4}$$

On $\rho = \alpha(1 + 3^{1/2}i)$ we have

$$|e^{-\omega_3\rho}\delta_1(\rho)| = |2e^{-3\alpha}\cos(3^{1/2}\alpha) + 1|;$$

$$|I_1| \le \frac{1}{12\alpha^2}3(2+m)\int_0^1 |r(t)|\, dt = \frac{1}{4\alpha^2}(2+m)r,$$

where

$$I_1 = \frac{1}{3\rho^2}\int_0^1 \{\exp[(\omega_1 - \omega_3)\rho(1-t)]$$

$$+ \exp[(\omega_2 - \omega_3)\rho(1-t)] + 1\}[A(t, \rho) + z(t, \rho)]r(t)\, dt;$$

$$|I_2| \le \frac{1}{6\alpha}3(2+m)\int_0^1 |p(t)|\, dt = \frac{1}{2\alpha}(2+m)p,$$

where

$$I_2 = \frac{1}{3\rho}\int_0^1 \{\exp[(\omega_1 - \omega_3)\rho(1 - t)]$$

$$- \omega_2 \exp[(\omega_2 - \omega_3)\rho(1 - t)] - \omega_3\}[A(t, \rho) + z(t, \rho)]p(t)\, dt;$$

$$| I_3 | \leq \frac{| p(1) |}{4\alpha^2}(1 + e^{-3\alpha} + m),$$

where

$$I_3 = p(1)[A(1, \rho) + z(1, \rho)]\rho^{-2}.$$

On $\rho = 2(n + 1)\pi/3^{1/2} + i\beta$, $\beta \geq 0$, we have

$$| e^{-\omega_3\rho}\delta_1(\rho) | = \left| \exp\left\{-\frac{3}{2}\left[2(n+1)\frac{\pi}{3^{1/2}} + i\beta\right] + \frac{3^{1/2}i}{2}\left[2(n+1)\frac{\pi}{3} + i\beta\right]\right\}\right.$$

$$\left. + \exp\left\{3^{1/2}i\left[2(n+1)\frac{\pi}{3^{1/2}} + i\beta\right]\right\} + 1 \right|$$

$$= \left| \exp[-3^{1/2}(n+1)\pi] \exp\left(-\frac{3}{2}i\beta\right)(-1)^n \exp\left(-\frac{3^{1/2}}{2}\beta\right)\right.$$

$$\left. + \exp(-3^{1/2}\beta) + 1 \right|,$$

hence $| e^{-\omega_3\rho}\delta_1(\rho) | \geq 1 - \exp(3^{1/2}\pi)$; also

$$| I_1 | \leq \frac{9}{4}\frac{(2 + m)r}{[2(n+1)\pi]^2 + 3\beta^2},$$

$$| I_2 | \leq \frac{3^{1/2}}{2}\frac{(2 + m)p}{\{[2(n+1)\pi]^2 + 3\beta^2\}^{1/2}},$$

$$| I_3 | \leq \frac{3\,| p(1) |\,(1 + \exp(-3^{1/2}\beta) + m)}{[2(n+1)\pi]^2 + 3\beta^2}.$$

Now $| E(\rho) | \leq | I_1 | + | I_2 | + | I_3 |$, and clearly for $| \rho |$ large enough there exists k such that for $n = k, k + 1, \cdots$, (1.4) holds. We have then $| \rho^2\delta_1(\rho) | > | \rho^2 e^{\omega_3\rho}E(\rho) |$ on C_n. Now it is easily verified that there is just one zero of $\delta_1(\rho)$ inside C_n. Hence, by Rouché's theorem, there is just one root of the characteristic equation inside C_n, and since these roots must occur in complex conjugate pairs, this root is real. This proves the theorem.

By a suitable modification of the choice of contours C_n in the proof of the preceding theorem, the following result may be obtained.

Theorem 2. *Let the zeros of* $\delta_1(\rho)$ *be denoted by* $\rho_n^{(0)}$ *such that* $\rho_0^{(0)} < \rho_1^{(0)} < \cdots$. *Let* $\delta > 0$ *be given, and define* R_n *to be the circle* $| \rho - \rho_n^{(0)} | \leq \delta$. *Then for* k *large enough, each root* ρ_n *of the characteristic equation of* (1) *such that* $R(\rho_n) > 2k\pi/3^{1/2}$ *lies inside one of the circles* R_n, $n = k, k + 1, \cdots$, *there being exactly one* ρ_n, *necessarily real, in each such* R_n.

The proof will be omitted.[4]

From the details of the proof of Theorem 1, it may be seen that for r, p, and $|p(1)|$ small enough, k may be chosen equal zero. The question of whether or not in this case all characteristic values of (1) are real, then, reduces to the question of whether or not all roots ρ_n of the characteristic equation for which $R(\rho_n) < 2\pi/3^{1/2}$ are real. By means of the following lemma, we obtain a lower bound on $|\lambda_n|$; e.g., $|\rho_n|$; and then by means of the subsequent theorem, apply the arguments of Theorem 1 to a supplementary contour which borders on C_0 and extends to the region for which no ρ_n can occur.

Lemma. *Let λ be a characteristic value of* (1). *Then* $|\lambda|^2 \geq 6 - (m_1 + m_2)$ where $|p(t)|^2 \leq m_1$, $|q(t)|^2 \leq m_2$.

Proof. Let $v(x) = u'(x)$. Then by (1)

$$v''(x) = -[p(x)u'(x) + q(x)u(x) + \lambda u(x)], \qquad v(0) = v'(1) = 0.$$

Hence

$$v(x) = \sum_n \frac{c_n + b_n + \lambda a_n}{\mu_n} \phi_n(x) \tag{1.5}$$

where $\phi_n(x)$ and μ_n are the characteristic functions and numbers respectively of the system

$$\phi''(x) + \mu\phi(x) = 0, \qquad \phi(0) = \phi'(1) = 0$$

and

$$a_n = \int_0^1 u(t)\phi_n(t)\,dt, \qquad b_n = \int_0^1 u(t)q(t)\phi_n(t)\,dt, \qquad c_n = \int_0^1 u'(t)p(t)\phi_n(t)\,dt.$$

Hence by (1.5) and integration from 0 to x, one obtains

$$u(x) = \sum_n \frac{c_n + b_n + \lambda a_n}{\mu_n} z_n(x) \qquad \text{where} \qquad z_n(x) = \int_0^x \phi_n(t)\,dt.$$

Multiplying by $\phi_k(x)$ and integrating from 0 to 1 one obtains

$$a_k = \int_0^1 \phi_k(x)u(x)\,dx = \sum_n \frac{c_n + b_n + \lambda a_n}{\mu_n} \int_0^1 z_n(x)\phi_k(x)\,dx.$$

Hence, using Schwarz's inequality,

$$|a_k|^2 \leq M \sum_n (|c_n|^2 + |b_n|^2 + |\lambda|^2 |a_n|^2) \left| \int_0^1 z_n(x)\phi_k(x)\,dx \right|^2$$

where

$$M = \sum_n \frac{1}{\mu_n^2} = \frac{1}{6}\,.$$

Summing over k and using Parseval's theorem:

$$\begin{aligned} \sum_k |a_k|^2 &\leq M \sum_n (|c_n|^2 + |b_n|^2 + |\lambda|^2 |a_n|^2) \sum_k \left| \int_0^1 z_n(x)\phi_k(x)\,dx \right|^2 \\ &= M \sum_n (|c_n|^2 + |b_n|^2 + |\lambda|^2 |a_n|^2) \int_0^1 |z_n(x)|^2\,dx. \end{aligned} \tag{1.6}$$

[4]See L. E. Ward, [4] pp. 419-420.

However, since

$$\int_0^1 | z_n(x) |^2 \, dx \leq \int_0^1 \left\{ \int_0^1 | \phi_n(t) |^2 \, dt \right\} dx = 1,$$

and since we may assume

$$\int_0^1 | u(x) |^2 \, dx = 1,$$

another application of Parseval's theorem to (1.6) gives

$$1 \leq M \left\{ \int_0^1 | p(t)u'(t) |^2 \, dt + \int_0^1 | q(t)u(t) |^2 \, dt + | \lambda |^2 \right\}. \tag{1.7}$$

Let $| p(t) |^2 \leq m_1$, $| q(t) |^2 \leq m_2$; then from (1.7) one has

$$1 \leq M \left\{ m_1 \int_0^1 | u'(t) |^2 \, dt + m_2 + | \lambda |^2 \right\}. \tag{1.8}$$

Multiplying (1.5) by $\phi_k(x)$ and integrating from 0 to 1, one obtains

$$\left| \int_0^1 \phi_k(x) u'(x) \, dx \right|^2 = \left| \frac{c_k + b_k + \lambda a_k}{\mu_k} \right|^2 \leq \frac{| c_k |^2 + | b_k |^2 + | \lambda |^2 \, | a_k |^2}{\mu_k^2},$$

and summing over k and applying Parseval's theorem again:

$$\int_0^1 | u'(x) |^2 \, dx \leq M \left(m_1 \int_0^1 | u'(x) |^2 \, dx + m_2 + | \lambda |^2 \right),$$

from which it follows that

$$(1 - Mm_1) \int_0^1 | u'(x) |^2 \, dx \leq M(m_2 + | \lambda |^2). \tag{1.9}$$

Hence, by (1.8) and (1.9):

$$(1 - Mm_1) \leq M[m_1 M(m_2 + | \lambda |^2) + (m_2 + | \lambda |^2)(1 - Mm_1)] = M(m_2 + | \lambda |^2).$$

From this one obtains

$$\frac{1 - M(m_1 + m_2)}{M} \leq | \lambda |^2.$$

Since $M = 1/6$, the proof of the lemma is complete.

Theorem 3. *Let p, r, m_1, m_2 be defined as in previous theorems and lemmas. Then if these constants and $| p(1) |$ are sufficiently small, all characteristic values of the system* (1) *are real.*

Proof. It has already been noted that for sufficiently small values of p, r, and $| p(1) |$, k may be chosen equal zero. We now consider the supplementary contour C_{-1}, a trapezoid whose bases are the segments:

$$R(\rho) = \frac{2\pi}{3^{1/2}}, \qquad | \arg \rho | \leq \frac{\pi}{3};$$

$$R(\rho) = \frac{R}{2}, \qquad | \arg \rho | \leq \frac{\pi}{3};$$

where $R = [6 - (m_1 + m_2)]^{1/6}$. By Theorem 1 and the lemma all non-real characteristic values $\lambda_n = \rho_n^3$ must be such that ρ_n is in or on C_{-1}. However, for sufficiently small p, r, m_1, m_2, and $|p(1)|$, the argument used to prove Theorem 1 is applicable to C_{-1} and we conclude that there is just one root ρ_{-1}, necessarily real, inside C_{-1}. This proves the theorem.

PART II

Consider the system

$$[f(t)y''(t)]' + \lambda g(t)y(t) = 0, \qquad y(0) = y'(0) = y''(1) = 0, \tag{2}$$

f and g real and analytic on $0 \leq t \leq 1$, $f(t) > 0$, $g(t) > 0$. The transformation

$$x = \frac{1}{d}\int_0^t h(s)\,ds \qquad \text{where} \qquad h(s) = \left[\frac{f(s)}{g(s)}\right]^{1/3}, \qquad d = \int_0^1 h(s)\,ds,$$

has an inverse $t = t(x)$. This inverse transforms (2) into

$$\begin{gathered} \frac{d^3}{dx^3}\,y[t(x)] + p_1(x)\,\frac{d^2}{dx^2}\,y[t(x)] + p_2(x)\,\frac{d}{dx}\,y[t(x)] + y[t(x)] = 0, \\ y(0) = y'(0) = y''(1) = 0, \end{gathered} \tag{2.1}$$

where

$$p_1(x) = \frac{1}{d}\left\{3h^4[t(x)]h'[t(x)] + \frac{f'[t(x)]}{g[t(x)]}\,h^2[t(x)]\right\}$$

$$p_2(x) = \frac{1}{d}\left\{h^3[t(x)]h''[t(x)] + \frac{f'[t(x)]}{f[t(x)]}\,h'[t(x)]\right\},$$

the notation $f'[t(x)]$ meaning that the substitution $t = t(x)$ has been made in the function of $t : f'(t)$.

Next, the substitution

$$y[t(x)] = u(x)\,\exp\left[-\frac{1}{3}\int_0^x p_1(s)\,ds\right]$$

and subsequent multiplication by

$$\exp\left[\frac{1}{3}\int_0^x p_1(s)\,ds\right]$$

reduces (2.1) to $u'''(x) + p(x)u'(x) + [q(x) + \lambda]u(x) = 0$, $u(0) = u'(0) = u''(1) = 0$, where

$$p(x) = p_2(x) - \frac{p_1^2(x)}{3} - p_1'(x),$$

$$q(x) = \frac{2}{27}\,p_1^3(x) - \frac{p_1''(x)}{3} - \frac{p_1(x)p_2(x)}{3}.$$

Hence the results of Part I apply to the system (2).

Noting the structure of p and q; e.g., p_1 and p_2, in terms of f and g, we state an obvious corollary of Theorem 3.

Corollary. *Let* $|f^{(n)}(t)| < \epsilon$, $|g^{(n)}(t)| < \epsilon$ for $n = 1, 2, 3, 4$. *Then for* ϵ *sufficiently small, all the characteristic values of* (2) *are real.*

It is also clear from the structure of p and q that less restrictive, although perhaps more complicated, conditions on f and g than those in the hypothesis of the above corollary will yield the same conclusion.

The author wishes to thank Professors W. Feller and W. R. Sears for suggesting the problem and for helpful suggestions toward its solution.

References

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[2] H. K. Cheng, *A variational method for the differential equation arising from aeroelastic problems*, Thesis, Graduate School of Aeronautical Engineering, Cornell University, Feb. 1950.

[3] L. E. Ward, *Some third order irregular boundary value problems*, Trans. Am. Math. Soc. **29**, 716-745 (1927).

[4] L. E. Ward, *A third order irregular boundary value problem and the associated series*, Trans. Am. Math. Soc. **34**, 417-434 (1932).

ON AN EQUATION OCCURRING IN THE HARMONIC ANALYSIS OF VISCOUS FLUID FLOW*

By RICHARD BELLMAN (*Stanford University*)

1. Introduction. It was shown by J. Kampé de Feriet[1] that the Fourier transform

$$z(w_1, w_2, t) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int \zeta(x, y) \exp[-i(w_1 x + w_2 y)]\, dx\, dy \tag{1}$$

of the vorticity, $\zeta(x, y)$, associated with the two-dimensional flow of an incompressible fluid extending over the entire (x,y)-plane, under mild conditions, satisfies the nonlinear integro-differential equation

$$\begin{aligned}\frac{\partial}{\partial t} z(w_1, w_2, t) &= -v(w_1^2 + w_2^2) z(w_1, w_2, t) \\ &\quad + 2 \int_{-\infty}^{\infty} \int \left(\frac{\theta_1 w_2 - \theta_2 w_1}{\theta_1^2 + \theta_2^2}\right) z(\theta_1, \theta_2, t) \bar{z}(\theta_1 + w_1, \theta_2 + w_2, t)\, d\theta_1\, d\theta_2,\end{aligned} \tag{2a}$$

and the boundary condition

$$z(w_1, w_2, 0) = \phi(w_1, w_2). \tag{2b}$$

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[1]J. Kampé de Feriet, *Harmonic analysis of the two-dimensional flow of an incompressible viscous fluid*, Q. Appl. Math. **6**, 1-13 (1948).

It seems quite difficult to determine the properties of the solution of 2a for general ϕ. Consequently, it may be of interest to indicate the following theorem which is an analogue of the well-known result of Poincaré and Liapounoff[2] concerning the stability of solutions of non-linear ordinary differential equations.

THEOREM: *If* $\mathrm{Max}_w \mid \phi(w_1, w_2) \mid$ *is sufficiently small, there is a solution to* (2a) *and* (2b) *which is unique, and satisfies the inequality*

$$\mid z(w_1, w_2, t) \mid \leq \frac{8 \,\mathrm{Max} \mid \phi(w_1, w_2) \mid}{[1 + v(w_1^2 + w_2^2)t]^2}, \tag{3}$$

for all w_1, w_2 *and* $t \geq 0$.

From this we conclude that the solution $z = 0$ *of* (2a) *is stable.*

By the expression "sufficiently small" we mean that there exists a constant $c = c(v)$ with the property that $\mathrm{Max} \mid \phi \mid \leq c$ suffices to establish (3). The constant c may be determined from the analysis below. However, we feel that the value of c obtained in the course of our proof has no particular significance. At the expense of decreasing c we can replace the exponent 2 on the right side of (3) by any arbitrary n.

While the general method, namely that of applying the technique of successive approximations, is standard, the details are not as simple as might be believed upon first viewing the equation. It might be expected that in place of (3) one could assert

$$\mid z(w_1, w_2, t) \mid \leq c_1 \,\mathrm{Max} \mid \phi(w_1, w_2) \mid \exp[-v(w_1^2 + w_2^2)t], \tag{4}$$

provided that, as above, $\mathrm{Max}_w \mid \phi(w_1, w_2) \mid$ is sufficiently small. This result seems difficult to prove, and it is quite possible that it is not true.

2. Proof of theorem. From (2) we obtain, assuming for the moment that the solution exists,

$$z(w_1, w_2, t) = \phi(w_1, w_2) \exp[-v(w_1^2 + w_2^2)t] + \int_0^t \exp[-v(w_1^2 + w_2^2)(t - t_1)] J(z) \, dt_1, \tag{1}$$

where we have set

$$J(z) = 2 \int_{-\infty}^{\infty} \int \frac{(\theta_1 w_2 - \theta_2 w_1)}{w_1^2 + w_2^2} \bar{z}(\theta_1 + w_1, \theta_2 + w_2, t) z(\theta_1, \theta_2, t) \, d\theta_1 \, d\theta_2 .$$

This equation is solved by the method of successive approximations, by means of the algorithm

$$\begin{aligned} z_0 &= \phi(w_1, w_2) \exp[-v(w_1^2 + w_2^2)t] \\ z_{n+1} &= z_0 + \int_0^t \exp[-v(w_1^2 + w_2^2)(t - t_1)] J(z_n) \, dt_1, \qquad n = 0, 1, \cdots . \end{aligned} \tag{3}$$

The first step of the proof consists of showing that the sequence $\{z_n\}$ is uniformly bounded by an appropriate function of w_1, w_2 and t, namely

$$\mid z_n \mid \leq \frac{8\alpha}{[1 + v(w_1^2 + w_2^2)t]^2}, \qquad n = 0, 1, \cdots ,$$

[2]R. Bellman, *On the boundedness of solutions of non-linear differential and difference equations*, Trans. Amer. Math. Soc. **62**, 357-86 (1947).

for all w_1, w_2, t, where we have set, for the sake of convenience,

$$\alpha = \operatorname*{Max}_{w} | \phi(w_1, w_2) |. \tag{5}$$

Throughout we shall use, without further mention, the following simple inequalities,

$$\begin{aligned} &\text{(a)} \quad e^{-x} \leq 1/(1 + x), \qquad x \geq 0 \\ &\text{(b)} \quad a/b \leq (1 + ax)/(1 + bx) \leq 1, \qquad b \geq a \geq 0, \qquad x \geq 0. \end{aligned} \tag{6}$$

We turn now to the proof of (4). The result is clear for $n = 0$, since

$$\begin{aligned} | z_0 | &\leq \operatorname*{Max}_{w} | \phi | \exp [-v(w_1^2 + w_2^2)t] \leq \frac{\alpha}{[1 + v/2(w_1 + w_2)t]^2} \\ &\leq 4\alpha/[1 + v(w_1 + w_2)t]^2. \end{aligned} \tag{7}$$

To treat the general case, we proceed by induction, assuming that (4) holds for $n = 0, 1, \cdots, N$, and then proving it for $N + 1$. We first require upper bounds for $J(z_N)$. Introducing polar coordinates,

$$\begin{aligned} \theta_1 &= R \cos \psi, \qquad w_1 = r \cos \theta \\ \theta_2 &= R \sin \psi, \qquad w_2 = r \sin \theta, \end{aligned} \tag{8}$$

$$J(z_N) = 2r \int_0^\infty \int_0^{2\pi} \sin (\psi - \theta) z_N(\theta_1, \theta_2, t)\bar{z}_N(\theta_1 + w_1, \theta_2 + w_2, t)\, dR\, d\psi. \tag{9}$$

Applying our inductive hypothesis,

$$\begin{aligned} | J(z_N) | &\leq 128\alpha^2 r \int_0^\infty \int_0^{2\pi} \left[\frac{1}{(1 + vR^2t)^2 \cdot [1 + vt(R^2 + 2Rr \cos (\theta - \psi) + r^2)]^2} \right] dR\, d\psi \\ &\leq c_1\alpha^2 r \int_0^\infty \left[\frac{dR}{(1 + vR^2t)^2[1 + v(R - r)^2t]^2} \right], \end{aligned} \tag{10}$$

where $c_1 = 256\pi$.

This last integral is now split into three parts,

$$c_1 r \int_0^\infty = c_1 r \int_0^{r/2} + c_1 r \int_{r/2}^{2r} + c_1 r \int_{2r}^\infty = J_1 + J_2 + J_3, \tag{11}$$

which we discuss separately. We have, since $r - R \geq r/2$ for $0 \leq R \leq r/2$,

$$\begin{aligned} J_1 &\leq \frac{c_1 r}{(1 + vr^2t/4)^2} \int_0^{r/2} \frac{dR}{(1 + vr^2t)^2} < \frac{16c_1 r}{(1 + vr^2t)^2} \int_0^\infty \frac{dR}{(1 + vR^2t)^2} \\ &< \frac{c_2 r}{t^{1/2}} \frac{1}{(1 + vr^2t)^2}, \end{aligned} \tag{12}$$

where

$$c_2 = 16c_1 \int_0^\infty ds/(1 + vs^2)^2.$$

Turning to J_2, we obtain

$$J_2 \leq \frac{3c_1}{2} r^2 \frac{1}{(1 + vr^2 t/4)^2} \leq \frac{24 c_1 r^2}{(1 + vr^2 t)^2} . \tag{13}$$

Finally, since $R - r \geq r$ for $2r \leq R < \infty$,

$$J_3 \leq \frac{c_1 M}{(1 + vr^2 t)^2} \int_{2r}^{\infty} \frac{dR}{(1 + vR^2 t)^2} < \frac{c_3 r}{t^{1/2}} \frac{1}{(1 + vr^2 t)^2} \tag{14}$$

where

$$c_3 = c_1 \int_0^{\infty} ds/(1 + vs^2)^2.$$

Collecting the results,

$$\begin{aligned} | J(z_N) | &\leq \alpha^2 (J_1 + J_2 + J_3) \\ &\leq \alpha^2 \left[\frac{(c_2 + c_3) r}{t^{1/2}} \frac{1}{(1 + vr^2 t)^2} + \frac{24 c_1 r^2}{(1 + vr^2 t)^2} \right]. \end{aligned} \tag{15}$$

Applying these inequalities to (3), the result is

$$\begin{aligned} | z_{N+1} | &\leq | z_0 | + \int_0^t \exp\,[-vr^2(t - t_1)] \,|\, J(z_N) \,|\, dt_1 \\ &\leq \frac{4\alpha}{(1 + vr^2 t)^2} + (c_2 + c_3)\alpha^2 r \int_0^t \frac{\exp\,[-vr^2(t - t_1)]\, dt_1}{(1 + vr^2 t_1)^2 (t_1)^{1/2}} \\ &\quad + 24 c_1 \alpha^2 r^2 \int_0^t \frac{\exp\,[-vr^2(t - t_1)]\, dt_1}{(1 + vr^2 t_1)^2} . \end{aligned} \tag{16}$$

The first integral may be written

$$r \int_0^t = r \int_0^{t/2} + r \int_{t/2}^t = I_1 + I_2 . \tag{17}$$

Then

$$\begin{aligned} I_1 &= r \int_0^{t/2} \frac{\exp\,[-vr^2(t - t_1)]}{(1 + vr^2 t_1)^2} \frac{dt_1}{(t_1)^{1/2}} \leq \exp\,[-vr^2 t/2] r \int_0^{t/2} \frac{dt_1}{(1 + vr^2 t_1)^2 (t_1)^{1/2}} \\ &\leq \frac{1}{(1 + vr^2 t/4)^2} r \int_0^{\infty} \frac{dt_1}{(1 + vr^2 t_1)^2 (t_1)^{1/2}} \leq \frac{c_4}{(1 + vr^2 t)^2} , \end{aligned} \tag{18}$$

where

$$c_4 = 16 \int_0^{\infty} ds/(1 + vs)^2 s^{1/2}.$$

The second integral leads to

$$I_2 = r \int_{t/2}^{t} \frac{\exp[-vr^2(t - t_1)]\, dt_1}{(1 + vr^2 t_1)^2 (t_1)^{1/2}} \leq \frac{r}{(1 + vr^2 t/2)^2} \int_{t/2}^{t} \exp[-vr^2(t - t_1)] \frac{dt_1}{(t_1)^{1/2}}$$

$$\leq \frac{4r}{(1 + vr^2 t)^2} \int_0^{t/2} \frac{\exp[-vr^2 t_1]\, dt_1}{(t - t_1)^{1/2}} \leq \frac{4r}{(1 + vr^2 t)^2} \int_0^{\infty} \frac{\exp[-vr^2 t_1]\, dt_1}{(t_1)^{1/2}} \tag{19}$$

$$\leq \frac{c_5}{(1 + vr^2 t)^2},$$

where

$$c_5 = 4 \int_0^{\infty} \frac{e^{-vs}\, ds}{s^{1/2}}.$$

The second integral in (16) is broken up in like fashion into I_3, the integral over $[0, t/2]$, and I_4, the integral over $[t/2, t]$. The first integral satisfies the inequality

$$I_3 \leq r^2 \exp[-vr^2 t/2] \int_0^{\infty} \frac{dt_1}{(1 + vr^2 t_1)^2} \leq \frac{c_6}{(1 + vr^2 t)^2}, \tag{20}$$

where

$$c_6 = 16 \int_0^{\infty} ds/(1 + vs)^2,$$

while the second satisfies the inequality

$$I_4 \leq c_7/(1 + vr^2 t)^2, \tag{21}$$

with $c_7 = 1/v$. Collating these results, we obtain

$$| z_{N+1} | \leq \frac{4\alpha}{(1 + vr^2 t)^2} + \frac{\alpha^2}{(1 + vr^2 t)^2} [(c_2 + c_3)(c_4 + c_5) + 24 c_1 (c_6 + c_7)]$$

$$\leq \frac{8\alpha}{(1 + vr^2 t)^2}, \tag{22}$$

provided that

$$\alpha \leq 2/[(c_2 + c_3)(c_4 + c_5) + 24 c_1 (c_6 + c_7)]. \tag{23}$$

This completes the induction.

We must now show that z_N converges to a solution of the original functional equation. In the usual manner, this is accomplished by demonstrating the uniform convergence of the series $\sum_{n=0}^{\infty} (z_{n+1} - z_n)$. From (2) we obtain

$$| z_1 - z_0 | \leq 2 \int_0^t \exp[-v(w_1^2 + w_2^2)(t - t_1)] \, | J(z_0) | \, dt_1 \leq \frac{c_8 \alpha^2}{(1 + vr^2 t)^2}. \tag{24}$$

It now follows by induction, using the same procedure as above, that there exists a constant c_9 such that

$$| z_{N+1} - z_N | \leq \frac{(c_9 \alpha)^{N+1}}{(1 + vr^2 t)^2}. \tag{25}$$

Hence if

$$\alpha \leq \text{Min}\ [1/c_9\ ,\ 1/(c_4 + c_5 + c_6 + c_7)], \tag{26}$$

we have uniform convergence of z_N to a function z over the entire $(w_1\ , w_2)$ plane and the infinite t-interval, $0 \leq t \leq \infty$. It follows from the uniform convergence that we can pass to the limit as $N \to \infty$ under the integral sign in (3), obtaining (1). Differentiation of (1) yields the original equation.

The uniqueness is now established in the standard fashion.

BOOK REVIEWS

Table of the Bessel functions $Y_0(z)$ and $Y_1(z)$ for complex arguments. Prepared by the Computation Laboratory, National Bureau of Standards. Columbia University Press, New York, 1950. xi + 427 pp. $7.50.

This volume supplements the earlier volume of tables of $J_0(z)$ and $J_1(z)$ for complex arguments [see Q. of Appl. Math. 2, 276 (1944) and **6**, 95 (1948)]. The main tables give $Y_0(\rho e^{i\varphi})$ and $Y_1(\rho e^{i\varphi})$ to ten decimal places for $\rho = 0(.01)10$ and $\varphi = 0(5°)90°$. Auxiliary tables give $Y_0(\rho e^{i\varphi}) - (2/\pi)J_0(\rho e^{i\varphi}) \log \rho$ and $Y_1(\rho e^{i\varphi}) - (2/\pi)J_1(\rho e^{i\varphi}) \log \rho + (2/\pi\rho)e^{-i\varphi}$, the complex zeros of Bessel functions, and five-point Lagrangian interpolation coefficients.

W. PRAGER

The inelastic behavior of engineering materials and structures. By Alfred M. Freudenthal. John Wiley & Sons, Inc., New York and Chapman & Hall, Limited, London, 1950. xvi +587 pp. $7.50.

The amazing scope of the book and its detailed coverage of so many facets of inelastic action bear eloquent testimony to the author's wide-spread reading and his own research. Quantum statistics, conventional metallurgy, mathematical theories of plasticity, visco-elasticity, stress analysis solutions, and design criteria for metals and concrete are all presented from a unified and extremely interesting point of view. The reader is made to feel equally familiar with electron clouds, simple and complex mechanical models of the behavior of real materials, Brownian motion, limit design, and testing machines.

The only objection to be noted is that little indication is given at the highly controversial nature of the field. Opinions are often stated as facts. For example, this reviewer believes that much of the material on thermodynamics and the mechanical equation of state is based on demonstrably over-simple and probably incorrect assumptions about the dissipated work. However, read with an open and skeptical mind the book is invaluable.

D. C. DRUCKER

Electromagnetic fields. Theory and application. Volume I: Mapping of Fields. By Ernst Weber. John Wiley & Sons, Inc., New York and Chapman & Hall Limited, London, 1950. xiv + 590 pp. $10.00.

The author has divided electromagnetic theory into static electric and magnetic fields on one hand

and dynamic electromagnetic fields on the other hand. This book (Volume I) deals with static electric and magnetic fields or in other words with the methods of potential theory. The list of contents: (1) The Electrostatic Field, (2) The Magnetostatic Field, (3) General Field Analogies, (4) Fields of Simple Geometries, (5) Experimental Mapping Methods, (6) Field Plotting Methods, (7) Two-Dimensional Analytic Solutions, (8) Three-Dimensional Analytic Solutions and Appendices.

The book is very carefully written throughout and the reviewer feels that particular attention should be called to Chapter 7 which contains an excellent discussion of conformal transformations and mapping. Chapter 6 on field plotting methods contains a rather thorough discussion of the method of images both electric and magnetic. Under numerical methods of field plotting the author includes a discussion of relaxation methods.

Altogether the reviewer feels that this book should be of real interest to a rather large group of physicists, engineers, and applied mathematicians.

ROHN TRUELL

The evolution of scientific thought. From Newton to Einstein. By A. d'Abro. Second edition, revised and enlarged. Dover Publications, Inc., 1950. xx + 481 pp. $3.95.

This "semi-popular book", using "non-technical language" (?) hopes to "serve as a general introduction" to the theory of relativity, "to whet the appetite for further knowledge" (the quotes are from Author's Preface). Its four parts deal with pre-relativity physics, the special theory of relativity, the general theory of relativity, and the methodology of science. We already have half-a-dozen short and masterly treatments of the same field by Einstein, Weyl and Eddington, and this one suffers badly in comparison. This very long book has seemed consistently boring to this reader. As an introduction to serious work it is far from helpful, as it contains no index, no bibliography, and quotations are made throughout without page reference. There are fifteen portraits of mathematicians (without indication of origin) but the pleasure the reader should have in looking at them is completely spoiled by fourteen of them being crowded together at the center of the book. It is to be hoped that this publisher's experiment, which no doubt made life easier for the make-up man, will not be repeated.

P. LE CORBEILLER

SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

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[illegible] should be taken to avoid [illegible] (1), the letter *l* and the prime ('), between [illegible]

[illegible] in exponents should be clearly indicated.

[illegible] should be entirely avoided because they require costly hand-composition [illegible] the latter should be used.

[illegible]

[illegible] any complicated expression that recurs frequently [illegible]

[illegible] the symbol exp should be used, particularly if such [illegible] Thus,

$$\exp [(a^2 + b^2)^{1/2}] \text{ is preferable to } e^{(a^2+b^2)^{1/2}}$$

[illegible] denominators of fractions should [illegible] Thus,

$$\frac{\cos (\pi x/2b)}{\cos (\pi a/2b)} \text{ is preferable to } \frac{\cos \frac{\pi x}{2b}}{\cos \frac{\pi a}{2b}}$$

In many instances the use of negative exponents permits saving of space. Thus,

$$\int u^{-1} \sin u \, du \text{ is preferable to } \int \frac{\sin u}{u} du.$$

[illegible] slight variations in [illegible] the order of symbols should therefore be carefully considered. Thus,

$$(a + bx) \cos t \text{ is preferable to } \cos t \, (a + bx).$$

[illegible] more widely than in print. [illegible] Thus,

$$[(a + b)^2 + \ldots] \text{ is preferable to } [\ldots]$$

Cuts: [illegible] It is recommended to [illegible] Drawings which are unsuitable for reproduction will be returned to the author [illegible] should be written on a separate sheet.

Bibliography: [illegible] at the end of the manuscript. References to the [illegible]

[illegible] vol. 2 [illegible] Note that the number of the series is not separated by [illegible] of the volume.

[illegible] language require this. Thus, [illegible] but the corresponding German title would [illegible]

[illegible] English translation added in [illegible]

Footnotes: [illegible] should be avoided. Footnotes containing mathematical formulas are not [illegible]

Abbreviations: [illegible] like Eq., Eqs., Fig., Sec., Art., etc. [illegible] at the first word of a sentence. [illegible]

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